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STATE SPACE APPLICATION TO SYSTEM DESIGN

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STATE SPACE APPLICATION  
TO SYSTEM DESIGN

by

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B.S., Naval Academy, 1961



Submitted in partial fulfillment of the  
requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL  
June 1968

1968

MOCK, S.

## ABSTRACT

The availability of digital and hybrid computers has led to the development of the state space approach and optimization theory for the analysis and design of control systems, particularly in space oriented problems where meaningful cost criteria can be defined. In this thesis optimization theory is investigated as applied to classical control systems, such as regulators, to determine if these techniques may be used in the design of systems to meet conventional performance standards. As part of this investigation a method has been developed which yields the overall state equations for a system from the state equations of the individual components. Also, since optimal designs are usually non-linear and time varying, a discussion of stability criteria for these systems is included.

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## INTRODUCTION

There are two basic approaches to the analysis and design of control systems. The classical approach uses a transfer function to describe the system. Since this is primarily a frequency (either  $s$  or  $\omega$ ) domain description, most of the design and analysis techniques are formulated in the frequency domain. The state space approach describes the system with a set of first order differential equations. Since this is primarily a time domain description it is logical that design and analysis techniques be formulated in the time domain. Part of the analysis problem consists of combining the individual component descriptions to form the overall system description. The methods for forming the system description with the classical transfer function are well established. However, for components described in the state space, such methods are not established. In Chapter I, a method is introduced which permits the engineer to obtain the system equations from the component state equations in a straightforward manner.

One class of design problems which uses the state space description is to be found in optimal control theory. Optimization and state space techniques can be applied to such problems as minimum time, minimum fuel, and trajectory calculations with much success. Their application to the design of servomechanisms and regulators is questionable. In Chapter II, optimal control theory is applied to the regulator problem in order to obtain a better understanding of the meaning of optimality. This investigation naturally leads to the question as to how the engineer can use this theory to design a regulator to meet specific design criteria.

Optimal control theory, more often than not, leads to non-linear and time varying systems. Stability is therefore an important consideration when using the state space approach. Since there is no simple stability criterion that is applicable to all non-linear and time varying systems, Lyapunov's second method must be used to develop criteria for specific classes of non-linearities. Lyapunov's second method is particularly useful in state space because it is formulated in the time domain. In Chapter III, stability criteria that exist for a particular class of time varying systems are investigated. This system is of particular interest because it can be the result of an optimal linear regulator design.

## CHAPTER I

### State Space Approach to Component Interconnections

1.1. Introduction. Since the transfer function is not a valid representation of a non-linear system, design tools such as root locus, Routh criterion, Nyquist criterion, and frequency response cannot be applied to non-linear systems. The state space description has a wider application because it is a valid representation of non-linear, linear, time invariant, and time varying systems. Often in the analysis and design problem, the internal behavior of the system is of concern. The transfer function is an input - output relationship and therefore provides no information about internal behavior. The state equations provide both input - output and internal information about the system. The state variables themselves determine this internal behavior. It is significant to note that the transfer function concept implies a single input-single output for each component or block of a system. The state space approach enables one to describe the system in a compact and simple form even though the system is composed of components which are multi-input and multi-output. The objective of this chapter is to present a simple method for obtaining the system state equations from the component state equations for component interconnections such as cascade, parallel, and feedback; and also to present a general method for the same task when the system is composed of many components which are interconnected in any conceivable way. The advantage of the two methods developed here is that the identity of the states of the individual components are preserved in the system description. Choate and Sage<sup>21</sup> have developed a procedure to yield the system state equations in phase variable form.

This is a mathematically convenient form but it does not preserve the identity of the individual component states in the system description.

## 1.2 State Equations and the Invariance of the Classical Transfer Function.

Any dynamic component of a control system is completely described by the state equations

$$\dot{\underline{x}} = \underline{A}\underline{x} + \underline{B}\underline{u} \quad (1-1)$$

$$\underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u} \quad (1-2)$$

where

$\underline{A}$  = an  $n \times n$  matrix

$\underline{B}$  = an  $n \times m$  matrix

$\underline{C}$  = an  $s \times n$  matrix

$\underline{D}$  = an  $s \times m$  matrix

$\dot{\underline{x}}$  = an  $n \times 1$  column vector of state derivatives

$\underline{x}$  = an  $n \times 1$  column vector of states

$\underline{u}$  = an  $m \times 1$  input vector of forcing functions

$\underline{y}$  = an  $s \times 1$  output vector

$s$  = the number of outputs

$n$  = the component order

$m$  = the number of inputs

An expanded form for Equations (1-1) and (1-2) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1n} \\ A_{21} & A_{22} & \cdot & \cdot & \cdot & A_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \cdot & \cdot & \cdot & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & \cdot & \cdot & \cdot & B_{1m} \\ B_{21} & B_{22} & \cdot & \cdot & \cdot & B_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{n1} & B_{n2} & \cdot & \cdot & \cdot & B_{nm} \end{bmatrix} \underline{u} \quad (1-3)$$

$$\underline{Y} = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & \cdot & C_{1n} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ C_{s1} & C_{s2} & \cdot & \cdot & \cdot & \cdot & C_{sn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & \cdot & \cdot & \cdot & \cdot & D_{1m} \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ D_{s1} & D_{s2} & \cdot & \cdot & \cdot & \cdot & D_{sm} \end{bmatrix} \underline{u} \quad (1-4)$$

where  $A_{ij}$ ,  $B_{ij}$ ,  $C_{ij}$ , and  $D_{ij}$  are obtained from the basic differential equations describing the component and thus may be constant, time varying, or functions of  $x$ , or both. Henceforth, Equations (1-3) and (1-4) will be referred to as the generic form.

Equations (1-1) and (1-2) have various special forms for which there are unique  $A$ ,  $B$ ,  $C$ , and  $D$  matrices depending upon the choice of state variables. In general, the choice of states is arbitrary and thus many representations exist. However, only certain specific choices have physical meaning (such as measureable indentifiable quantities) in any given system. It is worthwhile to demonstrate the invariance of a transfer function (single input/single output) to the manner in which the states are defined.<sup>3</sup> Let

$$\underline{Z} = T\underline{X} \quad (1-5)$$

Thus,  $\underline{Z}$  is a new set of state variables defined by means of the matrix transformation  $T$  operating on the given set of states  $\underline{X}$ . Equations (1-1) and (1-2) then become

$$T^{-1}\dot{\underline{Z}} = AT^{-1}\underline{Z} + B\underline{u} \quad (1-6)$$

$$\underline{Y} = CT^{-1}\underline{Z} + D\underline{u} \quad (1-7)$$

where  $\underline{u}$  and  $\underline{Y}$  are considered as column vectors (multiple input and output). Solving for the output, one obtains

$$Y(s) = C T^{-1} [T^{-1}s - AT^{-1}]^{-1} B \underline{u}(s) + D \underline{u}(s) \quad (1-8)$$

Equation (1-8) is equivalent to

$$\begin{aligned} Y(s) &= C [(T^{-1}s - AT^{-1})T]^{-1} B \underline{u}(s) + D \underline{u}(s) \\ &= C [sI - A]^{-1} B \underline{u}(s) + D \underline{u}(s) \end{aligned} \quad (1-9)$$

The output, Equation (1-9), is identical to that obtained from the original Equations (1-1) and (1-2). For the case of a single input and single output system, the transfer function becomes

$$\frac{Y(s)}{u(s)} = C [sI - A]^{-1} B + D \quad (1-10)$$

Equation (1-10) is independent of  $T$  and is the classical transfer function.

1.2.1 Special Forms. The following special forms are commonly used state representations.

(a) Normal Form. Phase Variables.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -A_{n1} & -A_{n2} & -A_{n3} & -A_{n4} & \cdots & -A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \quad (1-11)$$

$$Y = [C_{11} \ C_{12} \ C_{13} \ \cdots \ C_{1n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [D_{11}] u \quad (1-12)$$

This form corresponds to the following transfer function for single input-single output:

$$\frac{Y(s)}{U(s)} = D_{11} + \frac{C_{11} + C_{12}s + C_{13}s^2 + \dots + C_{1n}s^{n-1}}{A_{n1} + A_{n2}s + A_{n3}s^2 + \dots + A_{nn}s^{n-1} + s^n} \quad (1-13)$$

If the transfer function of a plant is expanded in the form of Equation (1-13), Equations (1-11) and (1-12) can be written directly provided the plant is controllable.<sup>3</sup> The normal form may also be obtained from the generic form by performing a matrix transformation such as that described by Equation (1-5). The procedure for selecting the T matrix that transforms the generic states into Normal Form (phase variables) is discussed by Browne.<sup>9</sup> The procedure is not straightforward and requires a great deal of mathematical manipulations. Thus, for a single input and single output component, it is usually simpler to go from the generic form to the transfer function and then to the Normal Form. This particular representation is of interest in Chapter II because it lends itself to certain mathematical conveniences not available in the other forms.

(b) Lur'e or Eigenvalue Expansion

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & \dots & 0 \\ 0 & A_{22} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \\ \vdots \\ B_{n1} \end{bmatrix} u \quad (1-14)$$

$$y = [1 \quad 1 \quad 1 \quad \dots \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [D_{11}] u \quad (1-15)$$

Equations (1-14) and (1-15) correspond to a component with simple poles only. For components with repeated poles, the A and B matrices are altered.<sup>1</sup> This Lur'e form corresponds to the following transfer function (for the component with simple poles only):

$$\frac{Y(s)}{X(s)} = D_{11} + \frac{B_{11}}{s-A_{11}} + \frac{B_{21}}{s-A_{22}} + \dots + \frac{B_{n1}}{s-A_{nn}} \quad (1-15a)$$

This state variable representation is obtained from the partial fraction expansion of the transfer function. However, it may also be obtained from the generic form by a proper choice of the T matrix.<sup>7</sup>

In practice, if the differential equation relating input to output is available, then the transfer function is easily obtained and the Normal form or Lur'e expansion follow readily. However, if the system is described by a sequence of differential equations then the generic form may be obtained easily. For multiple input and multiple output systems a scalar transfer function for the system cannot be described and therefore the generic form has to be utilized to obtain the special forms.

When components are interconnected it is possible to derive the differential equations for the combined system and then identify new variables and state equations. This involves a lot of unnecessary work and the original states lose their identity in the new system. Also, if the original component states are physically measureable quantities, this might not be true for the new states of the system. In the next section it is shown how the state equations for the system can be derived from the component state equations without changing the original states. The generic form will be used for simple interconnections.

1.3 Procedure for Obtaining System State Equations. Consider the state equations for two components each designated by a superscript within a parentheses.

$$\dot{\underline{X}}^{(1)} = A^{(1)} \underline{X}^{(1)} + B^{(1)} \underline{U}^{(1)} \quad (1-16)$$

$$\underline{Y}^{(1)} = C^{(1)} \underline{X}^{(1)} + D^{(1)} \underline{U}^{(1)} \quad (1-17)$$

$$\dot{\underline{X}}^{(2)} = A^{(2)} \underline{X}^{(2)} + B^{(2)} \underline{U}^{(2)} \quad (1-18)$$

$$\underline{Y}^{(2)} = C^{(2)} \underline{X}^{(2)} + D^{(2)} \underline{U}^{(2)} \quad (1-19)$$

Equations (1-16) through (1-19) can be combined into two matrix equations as follows:

$$\begin{bmatrix} \dot{\underline{X}}^{(1)} \\ \dot{\underline{X}}^{(2)} \end{bmatrix} = \begin{bmatrix} A^{(1)} & \underline{0} \\ \underline{0} & A^{(2)} \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} + \begin{bmatrix} B^{(1)} & \underline{0} \\ \underline{0} & B^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}^{(1)} \\ \underline{U}^{(2)} \end{bmatrix} \quad (1-20)$$

$$\begin{bmatrix} \underline{Y}^{(1)} \\ \underline{Y}^{(2)} \end{bmatrix} = \begin{bmatrix} C^{(1)} & \underline{0} \\ \underline{0} & C^{(2)} \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} + \begin{bmatrix} D^{(1)} & \underline{0} \\ \underline{0} & D^{(2)} \end{bmatrix} \begin{bmatrix} \underline{U}^{(1)} \\ \underline{U}^{(2)} \end{bmatrix} \quad (1-21)$$

Equations (1-20) and (1-21) imply that components one and two are operating independently, i.e., they are uncoupled. The next several sections investigate the effect of the following types of interconnections.

- (a) Cascade Combination
- (b) Parallel Combination
- (c) Feedback

Figure 1-1 is a block diagram representation of Equations (1-1) and (1-2) for a component. The double lines indicate that the signals are matrix quantities. Equations (1-1) and (1-2) can also be expressed in flow graph form as shown in Figure (1-2). The flowgraph form is used in the following development to obtain the system state equations.

1.3.1. Cascade Combination. When two components are cascaded, Figure 1-3 results. From Figure 1-3

$$\underline{U} = \underline{U}^{(1)} \quad (1-22)$$

$$\underline{Y} = \underline{Y}^{(2)} \quad (1-23)$$

$$\underline{Y}^{(1)} = \underline{U}^{(2)} \quad (1-24)$$

Substituting Equations (1-22) through (1-24) into Equations (1-16) through (1-19), one obtains the following new form for Equations (1-20) and (1-21):

$$\begin{bmatrix} \dot{\underline{X}}^{(1)} \\ \dot{\underline{X}}^{(2)} \end{bmatrix} = \begin{bmatrix} \underline{A}^{(1)} & \underline{Q} \\ \underline{B}^{(2)} \underline{C}^{(1)} & \underline{A}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} + \begin{bmatrix} \underline{B}^{(1)} \\ \underline{B}^{(2)} \underline{D}^{(1)} \end{bmatrix} \underline{U} \quad (1-25)$$

$$\underline{Y} = \begin{bmatrix} \underline{D}^{(2)} \underline{C}^{(1)} & \underline{C}^{(2)} \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} + \begin{bmatrix} \underline{D}^{(2)} \underline{D}^{(1)} \end{bmatrix} \underline{U} \quad (1-26)$$

The identical result is obtained by inspection from the matrix flowgraph of Figure 1-4, i.e., Equations (1-25) and (1-26) are obtained by writing the matrix equations of the new dependent variables as functions of the new independent variables.<sup>4</sup> An interesting modification to Equation (1-20), due to the cascade connection, is that the lower left 0 matrix is replaced by a coupling term. Thus, the lower left and upper right 0 matrix positions represent coupling of two components. Coupling connections are expressed in a general form in Figure 1-5. Note that the cascade combination is a special case of Figure 1-5. Cascade connection, when one of the two components is a matrix of gains, K, results in the following system state equations:

$$\begin{bmatrix} \dot{\underline{x}}^{(2)} \end{bmatrix} = \begin{bmatrix} A^{(2)} \end{bmatrix} \underline{x}^{(2)} + \begin{bmatrix} B^{(2)} K \end{bmatrix} \underline{u} \quad (1-27)$$

$$\begin{bmatrix} \underline{y}^{(2)} \end{bmatrix} = \begin{bmatrix} C^{(2)} \end{bmatrix} \underline{x}^{(2)} + \begin{bmatrix} D^{(2)} K \end{bmatrix} \underline{u} \quad (1-28)$$

1.3.2 Parallel Combination. When two components are connected in parallel, Figure 1-7 is obtained. From this figure

$$\underline{u} = \underline{u}^{(1)} = \underline{u}^{(2)} \quad (1-29)$$

$$\underline{y} = \underline{y}^{(1)} + \underline{y}^{(2)} \quad (1-30)$$

Substituting Equations (1-29) and (1-30) into Equations (1-16) through (1-19) yields the following new form for Equations (1-20) and (1-21):

$$\begin{bmatrix} \dot{\underline{x}}^{(1)} \\ \dot{\underline{x}}^{(2)} \end{bmatrix} = \begin{bmatrix} A^{(1)} & \underline{0} \\ \underline{0} & A^{(2)} \end{bmatrix} \begin{bmatrix} \underline{x}^{(1)} \\ \underline{x}^{(2)} \end{bmatrix} + \begin{bmatrix} B^{(1)} \\ B^{(2)} \end{bmatrix} \underline{u} \quad (1-31)$$

$$\underline{Y} = \begin{bmatrix} C^{(1)} & C^{(2)} \end{bmatrix} \begin{bmatrix} \underline{X}^{(1)} \\ \underline{X}^{(2)} \end{bmatrix} + \begin{bmatrix} D^{(1)} + D^{(2)} \end{bmatrix} \underline{u} \quad (1-32)$$

The same results are obtained by inspection of the flowgraph of Figure 1-8. As one would expect from paralleling two components, the output is merely the sum of the outputs of the two components operating independently.

**1.3.3 State Variable Feedback.** Introduction of state variable feedback to a single component is shown in Figure 1-9. From this figure

$$\underline{u}^{(1)} = \underline{u} - H \underline{X}^{(1)} \quad (1-33)$$

where H is a 1xn row vector of gains. Combining Equations (1-33), (1-16), and (1-17), one obtains the following results:

$$\dot{\underline{X}}^{(1)} = \begin{bmatrix} A^{(1)} & - B^{(1)} H \end{bmatrix} \underline{X} + B^{(1)} \underline{u} \quad (1-34)$$

$$\underline{Y} = \begin{bmatrix} C^{(1)} & - D^{(1)} H \end{bmatrix} \underline{X} + D^{(1)} \underline{u} \quad (1-35)$$

Using the flowgraph method and Figure 1-10, the same results are obtained by inspection. Introduction of state variable feedback changes the A and C matrices and thus enables one to move the open loop poles of the component to any desired location as shown by the system transfer function (for a single input and single output system).

$$\frac{Y(s)}{u(s)} = C \left[ sI - A \right]^{-1} B + D \quad (1-36)$$

where

$$C = C^{(1)} - D^{(1)}_H \quad (1-36a)$$

$$A = A^{(1)} - B^{(1)}_H \quad (1-36b)$$

$$B = B^{(1)} \quad (1-36c)$$

$$D = D^{(1)} \quad (1-36d)$$

Unity feedback is a special case of state variable feedback and therefore treated the same. To demonstrate the use of the above, several examples are considered. Appendix A gives a set of state equations for a few networks and machines. Normally the states of a component are arbitrary; but, for the components listed in Appendix A the states chosen are physically measureable quantities.

Example 1.1 Figure 1-11 is an example of a system formed by cascading several components. The state equations for the lead network (Appendix A) are

$$\begin{bmatrix} \dot{V}_c \end{bmatrix} = \begin{bmatrix} -\frac{R_1 + R_2}{R_1 R_2 C} \end{bmatrix} V_c + \begin{bmatrix} \frac{1}{R_2 C} \end{bmatrix} E_{in} \quad (1-37)$$

$$\begin{bmatrix} E_o \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix} V_c + \begin{bmatrix} 1 \end{bmatrix} E_{in} \quad (1-38)$$

The state equations describing the servomotor are

$$\begin{bmatrix} \dot{\Theta}_m \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{K_n + f}{J} \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_e}{J} \end{bmatrix} E_m \quad (1-39)$$

$$\Theta_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} \quad (1-40)$$

Using Equations (1-25) and (1-26), the combined system state equations can be written directly.

$$\begin{bmatrix} \dot{V}_c \\ \dot{\Theta}_m \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R_1+R_2}{R_1 R_2 C} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{K_e}{J} & 0 & -\frac{K_n+f}{J} \end{bmatrix} \begin{bmatrix} V_c \\ \Theta_m \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} \frac{K}{R_2 C} \\ 0 \\ \frac{K K_e}{J} \end{bmatrix} V_i \quad (1-41)$$

$$\Theta_L = \begin{bmatrix} 0 & T & 0 \end{bmatrix} \begin{bmatrix} V_c \\ \Theta_m \\ \dot{\Theta}_m \end{bmatrix} \quad (1-42)$$

T is the gearbox ratio. The important result here is that system state equations are written directly without using the transfer function of the components and therefore retaining the original component states.

Example 1.2 Unity feedback is now added to the system of Figure 1-11, and the resulting system is Figure 1-12. This type of feedback is the same as state variable feedback with a feedback gain matrix

$$H = \begin{bmatrix} 0 & T & 0 \end{bmatrix} \quad (1-43)$$

Therefore, using Equations (1-34) and (1-35), the system state equations are

$$\begin{bmatrix} \dot{V}_c \\ \dot{\Theta}_m \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R_1+R_2}{R_1 R_2 C} & -\frac{KT}{R_2 C} & 0 \\ 0 & 0 & 1 \\ -\frac{K_e}{J} & -\frac{K K_e T}{J} & -\frac{K_n+f}{J} \end{bmatrix} \begin{bmatrix} V_c \\ \Theta_m \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} \frac{K}{R_2 C} \\ 0 \\ \frac{K K_e}{J} \end{bmatrix} E \quad (1-44)$$

The examples above illustrate the simplicity in obtaining the system state equations from the component state equations. The very same results are obtainable through the use of transfer functions but with the added difficulty of identifying the original states. The general equations used to obtain the system state equations do not have to be remembered because they are easily obtained from the flowgraph method.

1.3.4. Complex Combination. More complex control systems often have component interconnections that are not entirely cascade, parallel, or feedback connections. The overall system may consist of components which are multi-input/multi-output, single input/single output, or combinations of both. Figure 1-13 is an example of a complex combination. The state equations for the overall system can be obtained by the flowgraph method. However, as the number of inputs and outputs of a component increases the flowgraph method becomes impractical. A method is proposed which will enable one to obtain the system state equations for any complex system as well as the cascade, parallel, and feedback connected systems. The general formulas for computing the system state equations are now derived.

If the state equations of the individual components are combined into one large matrix of equations so that the overall system inputs are at the top of the input vector  $\underline{u}$  and all other internal inputs form the bottom of the  $\underline{u}$  vector, then the combined matrix of state equations can be written as

$$\dot{\underline{X}} = \underline{A}\underline{X} + \underline{B}_a \underline{u}_a + \underline{B}_b \underline{u}_b \quad (1-46)$$

$\underline{u}_a$  is the vector of external system inputs and  $\underline{u}_b$  is the vector of internal component inputs. The same arrangement of the input vector allows the combined matrix of output equations to be written as

$$\underline{Y} = \underline{C} \underline{X} + \underline{D}_a \underline{u}_a + \underline{D}_b \underline{u}_b \quad (1-47)$$

From the diagram of the system, the  $\underline{u}_b$  vector can be written in terms of the  $\underline{Y}$  vector of the combined system. This relation is

$$\underline{u}_b = \underline{\Delta} \underline{Y} \quad (1-48)$$

The elements of  $\underline{\Delta}$ , the interconnection matrix, are determined by inspection. To illustrate the procedure of writing the combined matrix of state equations, output equations, and the interconnection matrix, consider the system of Figure 1-13. For this example there are only two system inputs,  $u_1^{(1)}$  and  $u_2^{(3)}$ . For simplicity, consider each component to have only one state (first order components). Equations (1-46) through (1-48) become

$$\begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_1^{(2)} \\ \dot{x}_1^{(3)} \\ \dot{x}_1^{(4)} \end{bmatrix} = \begin{bmatrix} A^{(1)} & 0 & 0 & 0 \\ 0 & A^{(2)} & 0 & 0 \\ 0 & 0 & A^{(3)} & 0 \\ 0 & 0 & 0 & A^{(4)} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \\ x_1^{(4)} \end{bmatrix} + \begin{bmatrix} B_1^{(1)} & 0 & B_2^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1^{(2)} & B_2^{(2)} & 0 & 0 & 0 \\ 0 & B_2^{(3)} & 0 & 0 & 0 & B_1^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_1^{(4)} & B_2^{(4)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(3)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(4)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} \quad (1-49)$$

$$\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_1^{(2)} \\ y_2^{(2)} \\ y_1^{(3)} \\ y_2^{(3)} \\ y_1^{(4)} \\ y_2^{(4)} \end{bmatrix} = \begin{bmatrix} C_1^{(1)} & 0 & 0 & 0 \\ C_2^{(1)} & 0 & 0 & 0 \\ 0 & C_1^{(2)} & 0 & 0 \\ 0 & C_2^{(2)} & 0 & 0 \\ 0 & 0 & C_1^{(3)} & 0 \\ 0 & 0 & 0 & C_2^{(3)} \\ 0 & 0 & 0 & C_1^{(4)} \\ 0 & 0 & 0 & C_2^{(4)} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \\ x_1^{(4)} \end{bmatrix} + \begin{bmatrix} D_{11}^{(1)} & 0 & D_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ D_{21}^{(1)} & 0 & D_{22}^{(1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11}^{(2)} & D_{12}^{(2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{21}^{(2)} & D_{22}^{(2)} & 0 & 0 & 0 \\ 0 & D_{11}^{(3)} & 0 & 0 & 0 & D_{11}^{(3)} & 0 & 0 \\ 0 & D_{21}^{(3)} & 0 & 0 & 0 & D_{21}^{(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_{11}^{(4)} & D_{12}^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & D_{21}^{(4)} & D_{22}^{(4)} \end{bmatrix} \begin{bmatrix} \underline{a}_1^{(1)} \\ \underline{a}_2^{(1)} \\ \underline{a}_1^{(2)} \\ \underline{a}_2^{(2)} \\ \underline{a}_1^{(3)} \\ \underline{a}_2^{(3)} \\ \underline{a}_1^{(4)} \\ \underline{a}_2^{(4)} \end{bmatrix} \quad (1-50)$$

$$\begin{bmatrix} \underline{a}_2^{(1)} \\ \underline{a}_1^{(2)} \\ \underline{a}_2^{(2)} \\ \underline{a}_1^{(3)} \\ \underline{a}_1^{(4)} \\ \underline{a}_2^{(4)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_1^{(2)} \\ y_2^{(2)} \\ y_1^{(3)} \\ y_2^{(4)} \end{bmatrix} \quad (1-51)$$

If Equation (1-48) is solved for  $\underline{y}$  and this result equated to Equation (1-47),  $\underline{a}_b$  can be obtained in terms of  $\underline{x}$  and  $\underline{a}_a$ . Substituting this result for  $\underline{a}_b$  into Equations (1-46) and (1-47) yields

$$\dot{\underline{x}} = [A + B_b(I - \Delta D_b)^{-1} \Delta C] \underline{x} + [B_a + B_b(I - \Delta D_b)^{-1} \Delta D_a] \underline{a}_a \quad (1-52)$$

$$\underline{y} = [C + D_b(I - \Delta D_b)^{-1} \Delta C] \underline{x} + [D_a + D_b(I - \Delta D_b)^{-1} \Delta D_a] \underline{a}_a \quad (1-53)$$

The complete derivation of Equations (1-52) and (1-53) is given in Appendix E. The above equations represent the state and output equations for any system. To illustrate the application of the above results,

consider the following examples. In each of these examples the elements of the A, B, C, and D matrices of the component state equations are arbitrarily chosen for illustration purposes only.

Example 1.3 The combined matrix state equations and output equations for the system of Figure 1-14 are

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \mathcal{L}_1^{(1)} \\ \mathcal{L}_2^{(1)} \\ \mathcal{L}^{(2)} \\ \mathcal{L}^{(3)} \end{bmatrix} \quad (1-54)$$

$$\begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{L}_1^{(1)} \\ \mathcal{L}_2^{(1)} \\ \mathcal{L}^{(2)} \\ \mathcal{L}^{(3)} \end{bmatrix} \quad (1-55)$$

The interconnection matrix equations are

$$\begin{bmatrix} \mathcal{L}^{(2)} \\ \mathcal{L}^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} \quad (1-56)$$

Therefore

$$B_a = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (1-57)$$

$$B_b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \quad (1-58)$$

$$D_a = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (1-59)$$

$$D_b = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix} \quad (1-60)$$

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (1-61)$$

Application of Equations (1-52) and (1-53) yields

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} \mathcal{N}_1^{(1)} \\ \mathcal{N}_2^{(1)} \end{bmatrix} \quad (1-62)$$

$$\begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 4 & 8 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \mathcal{N}_1^{(1)} \\ \mathcal{N}_2^{(1)} \end{bmatrix} \quad (1-63)$$

**Example 1.4** The combined state equations and output equations for the system of Figure 1-15 are

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \mathcal{N}_1^{(1)} \\ \mathcal{N}_2^{(2)} \\ \mathcal{N}_1^{(2)} \\ \mathcal{N}^{(3)} \end{bmatrix} \quad (1-64)$$

$$\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r^{(1)} \\ r_2^{(2)} \\ r_1^{(2)} \\ r^{(3)} \end{bmatrix} \quad (1-65)$$

$$\begin{bmatrix} r_1^{(2)} \\ r^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} \quad (1-66)$$

Again, application of Equations (1-52) and (1-53) yields

$$\begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_1^{(2)} \\ \dot{x}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ 8 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} r^{(1)} \\ r^{(2)} \end{bmatrix} \quad (1-67)$$

$$\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} r^{(1)} \\ r_2^{(2)} \end{bmatrix} \quad (1-68)$$

Example 1.5. Figure 1-16 is different from the previous two examples because a feedback loop is present. From flowgraph theory one would

suspect that the determinant of  $(I - \Delta D_b)$  can have a value different from unity. This is indeed the case. By inspection of the block diagram of the system for a closed loop, the necessity for computing the determinant of  $(I - \Delta D_b)$  can be established. Figure 1-16 has a closed loop provided  $y_2^{(1)}$  is a function of  $\lambda_2^{(1)}$  and  $y^{(3)}$  is a function of  $\lambda^{(3)}$ . The combined state equations for this system are

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \lambda_1^{(1)} \\ \lambda_2^{(1)} \\ \lambda^{(2)} \\ \lambda^{(3)} \end{bmatrix} \quad (1-69)$$

$$\begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1^{(1)} \\ \lambda_2^{(1)} \\ \lambda^{(2)} \\ \lambda^{(3)} \end{bmatrix} \quad (1-70)$$

$$\begin{bmatrix} \lambda_2^{(1)} \\ \lambda^{(2)} \\ \lambda^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} \quad (1-71)$$

Substituting into Equations (1-52) and (1-53) yields

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} -5 & 0 & -10 \\ -35 & 2 & -50 \\ -8 & 0 & -16 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} -7 \\ -35 \\ -8 \end{bmatrix} \begin{bmatrix} \lambda_1^{(1)} \end{bmatrix} \quad (1-72)$$

$$\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} = \begin{bmatrix} -7 & 0 & -10 \\ -2 & 0 & -5 \\ -21 & 4 & -30 \\ -4 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} -7 \\ -2 \\ -21 \\ -4 \end{bmatrix} \begin{bmatrix} r_1^{(1)} \end{bmatrix} \quad (1-73)$$

**Example 1.6** This example demonstrates the procedure for obtaining the system state equations when the components are connected by pure gains and summers. Consider the system of Figure 1-17. The combined state and output equations are

$$\begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_1^{(2)} \\ \dot{x}_1^{(3)} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r^{(2)} \\ r_1^{(3)} \\ r_2^{(3)} \end{bmatrix} \quad (1-74)$$

$$\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r^{(2)} \\ r_1^{(3)} \\ r_2^{(3)} \end{bmatrix} \quad (1-75)$$

For this system the interconnection matrix is

$$\begin{bmatrix} r^{(2)} \\ r_1^{(3)} \\ r_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y^{(2)} \\ y^{(3)} \end{bmatrix} \quad (1-76)$$

Again, application of Equations (1-52) and (1-53) yields the following state and output equations:

$$\begin{bmatrix} \dot{X}_1^{(1)} \\ \dot{X}_1^{(2)} \\ \dot{X}_1^{(3)} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 10 & -2 & 0 \\ 21 & 15 & -3 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 30 & 20 \\ 73 & 40 \end{bmatrix} \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \end{bmatrix} \quad (1-77)$$

$$\begin{bmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ Y^{(2)} \\ Y^{(3)} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 3 & 4 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_1^{(2)} \\ X_1^{(3)} \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 3 & 2 \\ 15 & 10 \\ 16 & 15 \end{bmatrix} \begin{bmatrix} r_1^{(1)} \\ r_2^{(1)} \end{bmatrix} \quad (1-78)$$

The examples above demonstrate the usefulness of the general method to a system composed of multi-input/multi-output components which are connected in any complicated fashion. For systems composed of many components, the method is particularly suitable to the digital computer.

In this chapter a brief introduction to the state variable description of a dynamic system is presented, and some general methods are presented for obtaining the state equations for the system from the component state equations. This system description lends itself to a time domain design procedure. The classical approach to the design of a regulator or servomechanism uses the transfer function description. The question now arises as to how one designs a system in the time domain to meet required specifications? This is discussed in Chapter II.

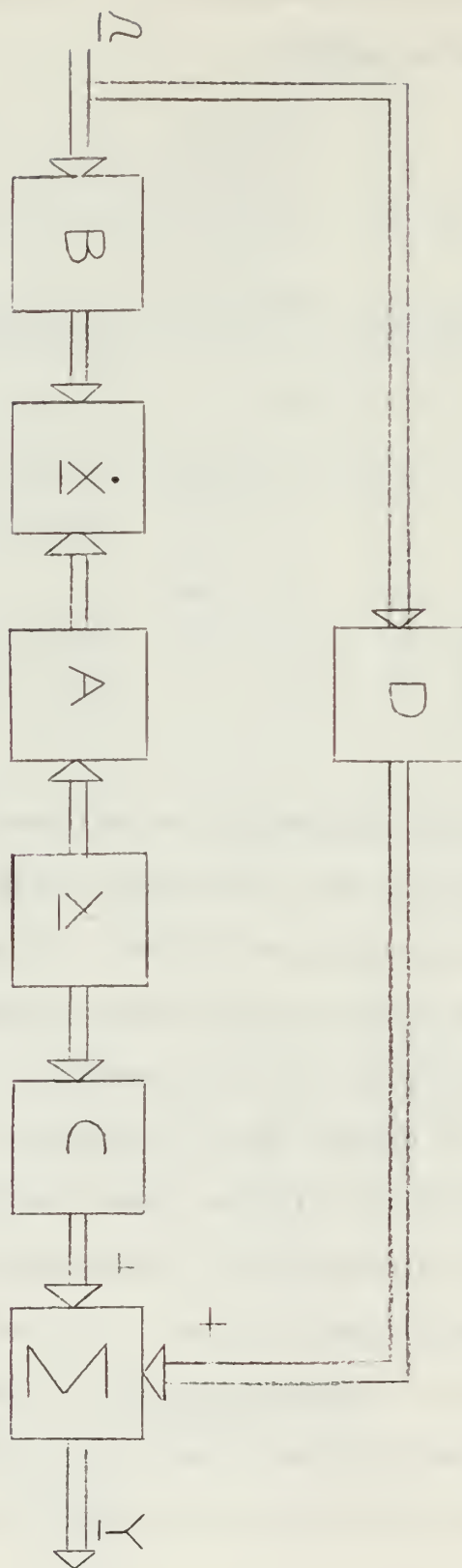


FIGURE 1-1. BLOCK DIAGRAM OF STATE EQUATIONS

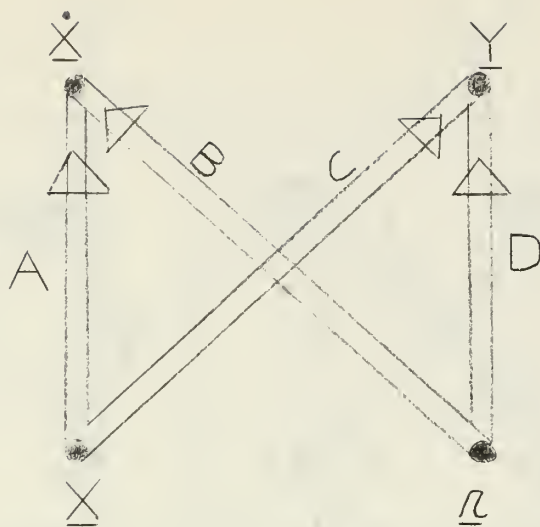


Figure 1-2. Flowgraph of State Equations

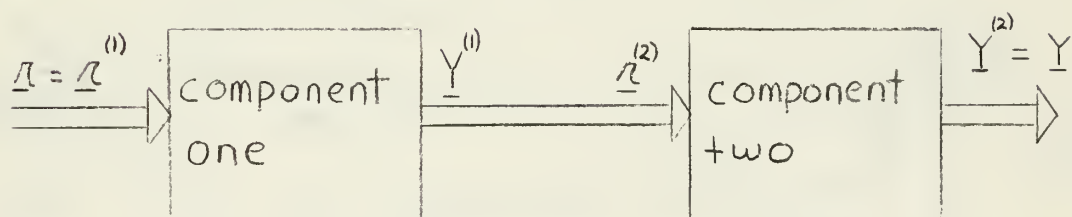


Figure 1-3. Cascade Connection of Two Components

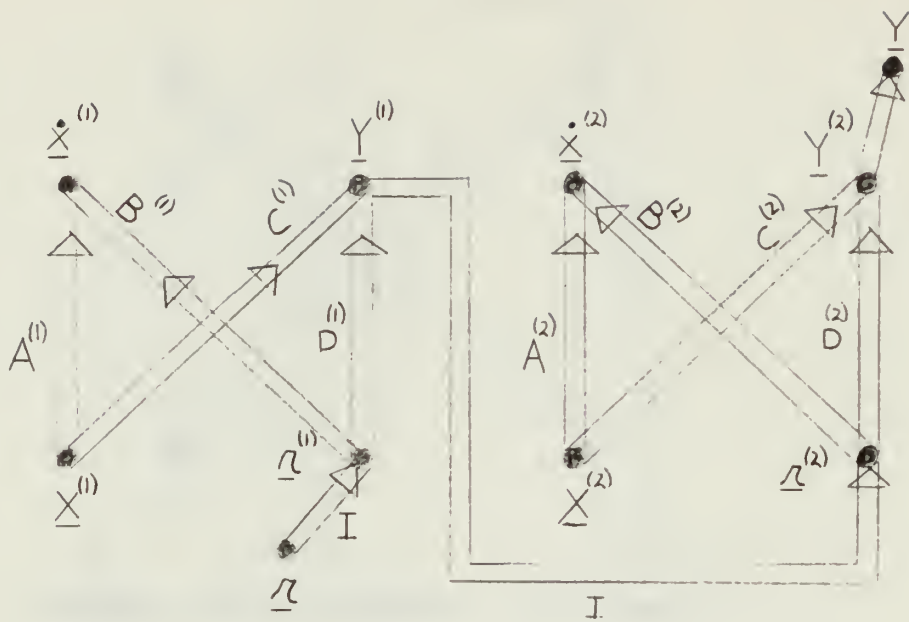


Figure 1-4. Flowgraph for Cascade Combination  
(I is the Identity Matrix)

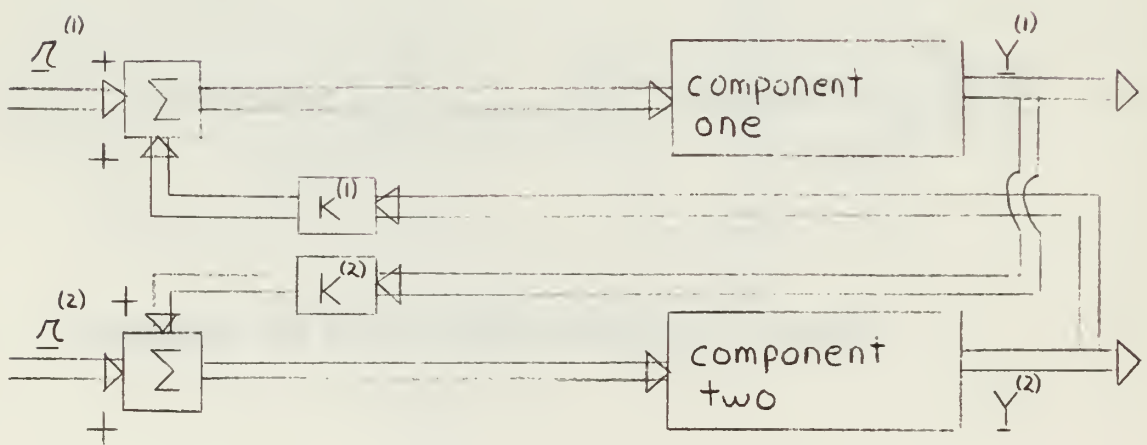


Figure 1-5. Block Diagram of Coupling

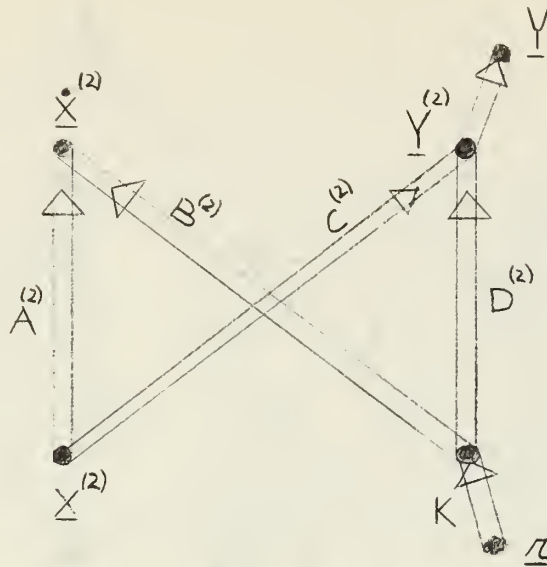


Figure 1-6. Flowgraph for a Cascaded Pure Gain Component

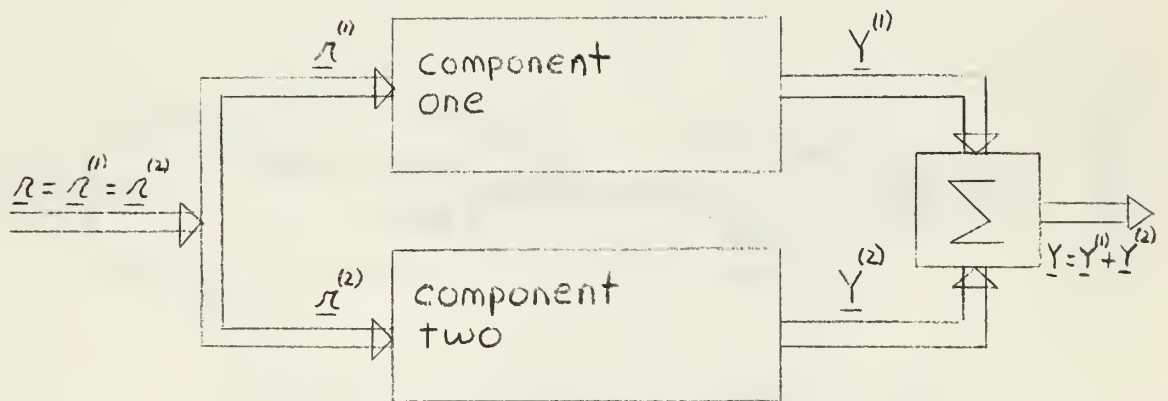


Figure 1-7. Block Diagram of a Parallel Combination

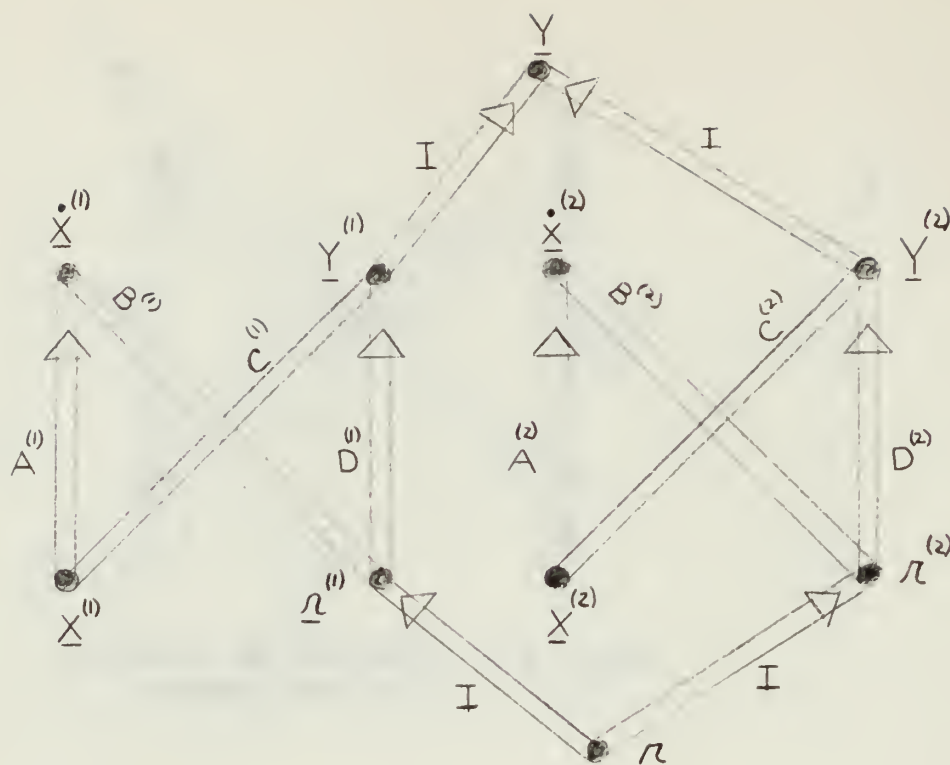


Figure 1-8. Flowgraph for a Parallel Combination

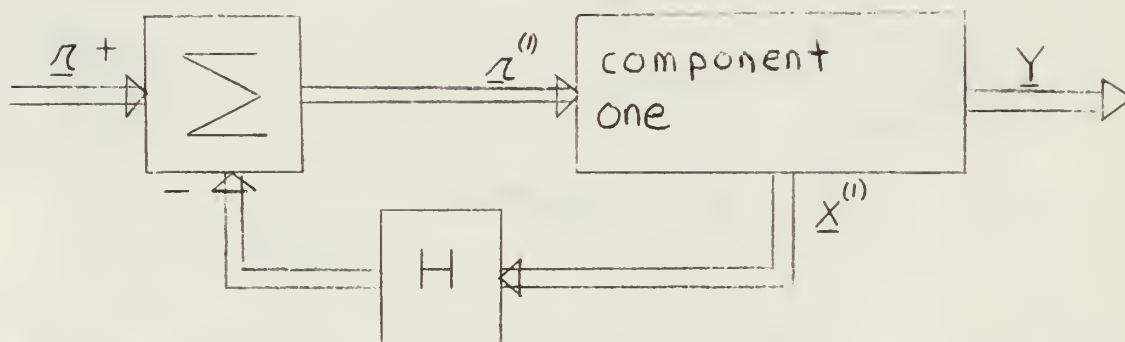


Figure 1-9. Block Diagram for State Feedback

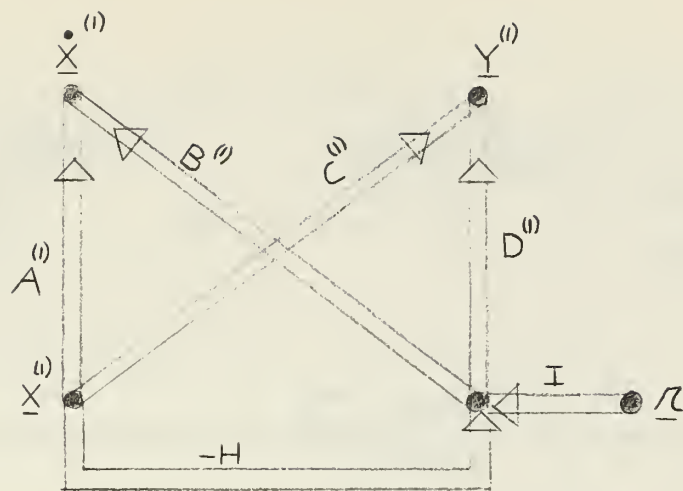


Figure 1-10. Flowgraph for State Feedback

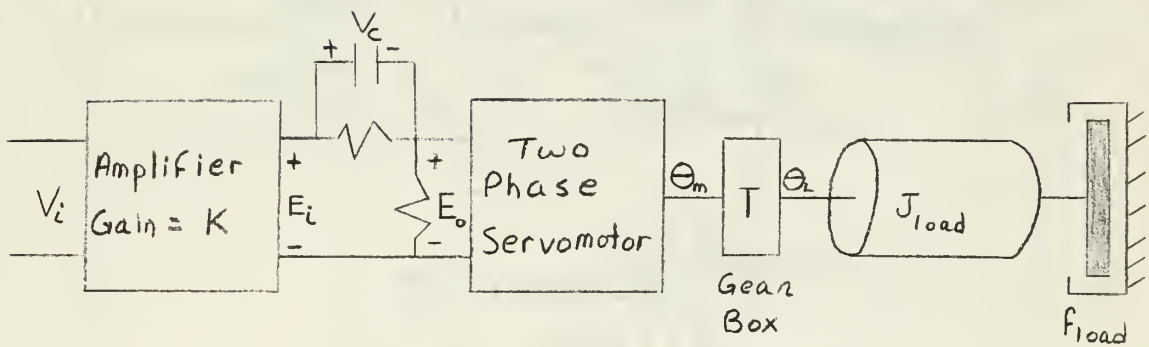


Figure 1-11. Block Diagram of Cascaded System

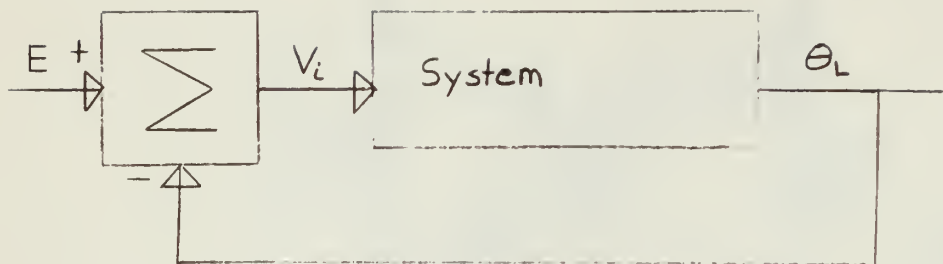


Figure 1-11. Block Diagram of Unity Feedback

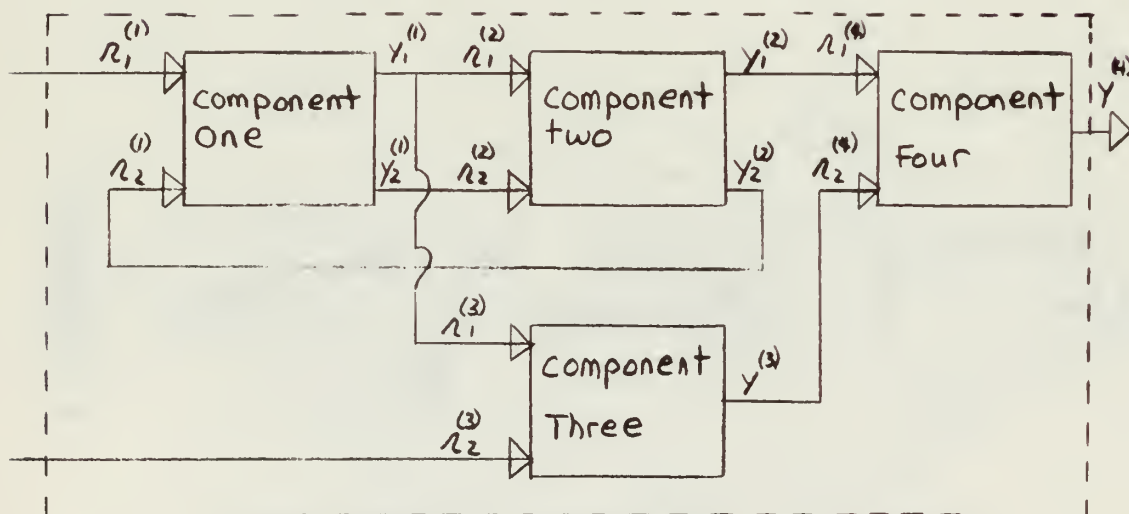


Figure 1-13. Complex Combination

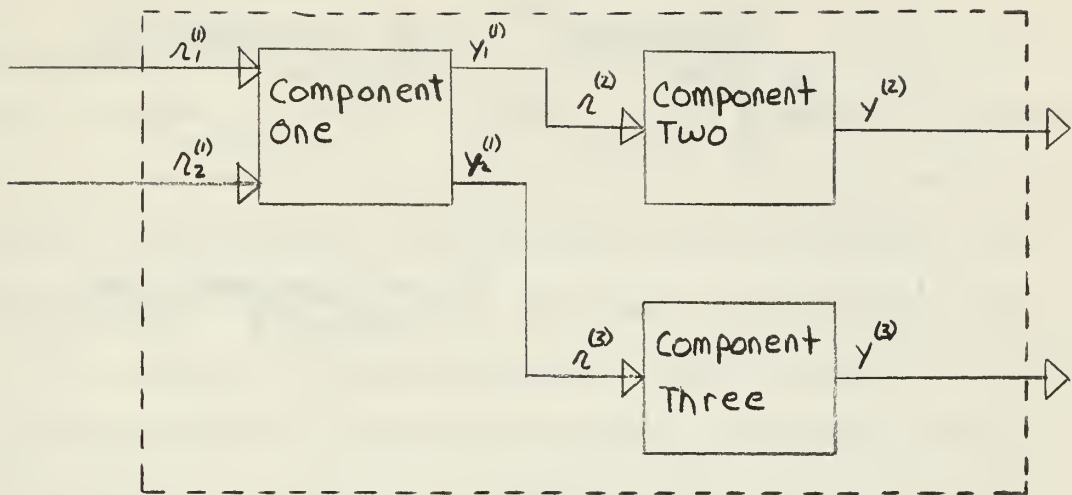


Figure 1-14. Example 1.3

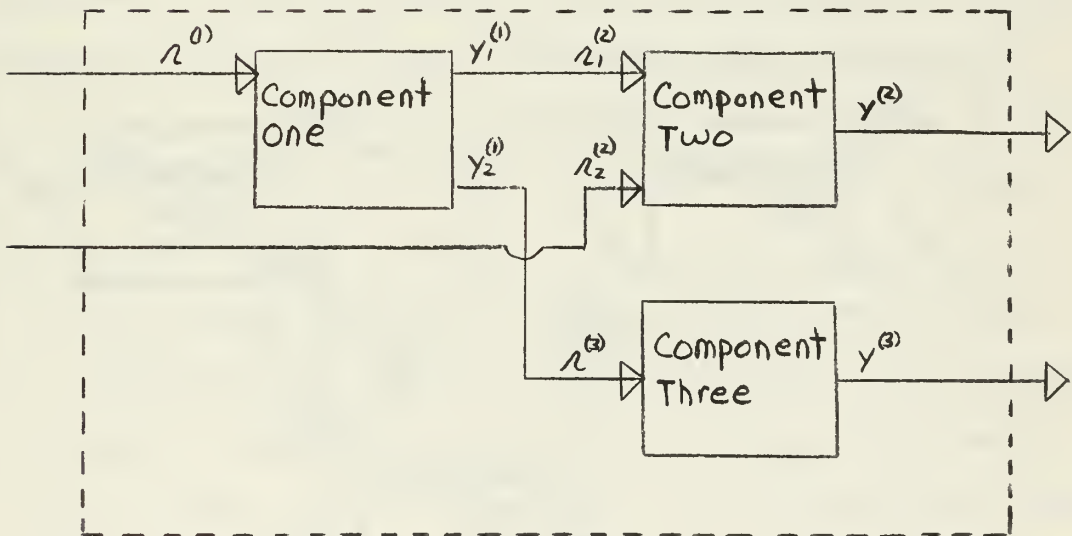


Figure 1-15. Example 1.4

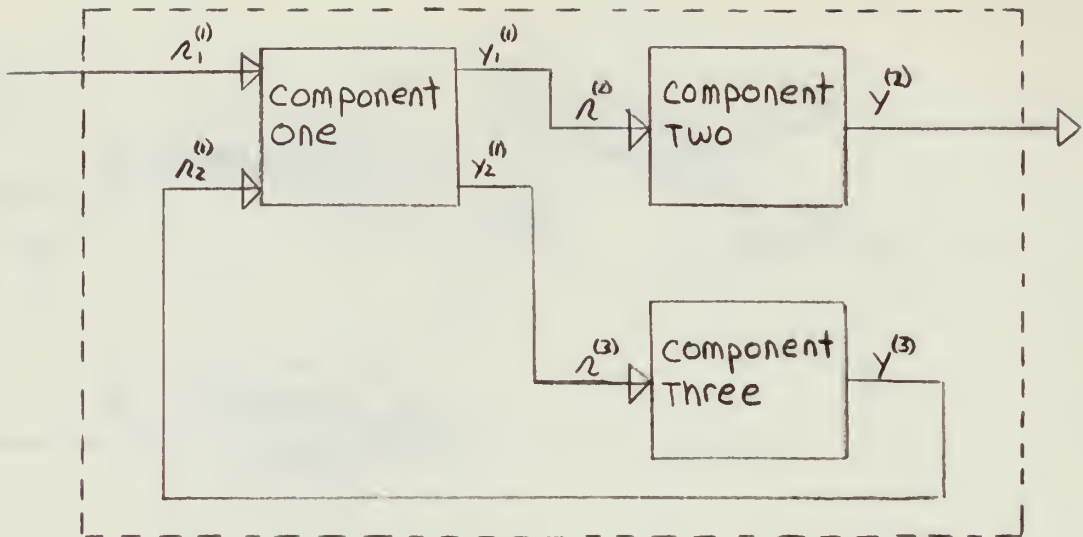


Figure 1-16. Example 1.5

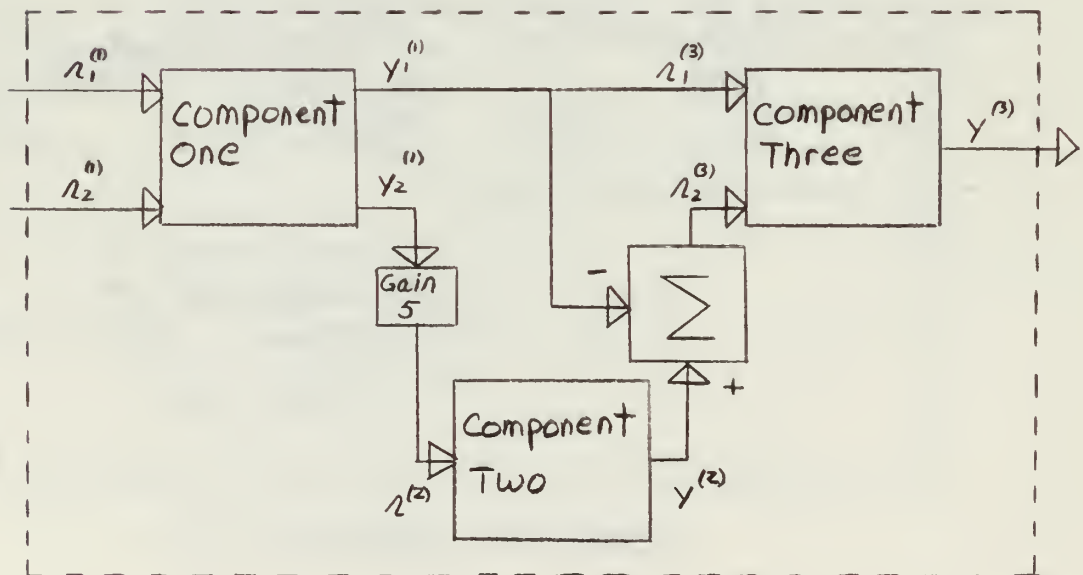


Figure 1-17. Example 1.6

## CHAPTER II

### System Design Using Optimization Theory

2.1. Introduction. In Chapter I the state space description was used to represent the control system. A design technique or approach which is based on this type representation is known as optimal control theory. Even though some of the optimal criteria are stated in the frequency domain, most are in the time domain. Kalman<sup>3</sup> argues that this is a weakness of the modern approach because the majority of the control concepts are expressed more simply in the frequency domain. The experienced designer of a servomechanism or regulator uses frequency domain concepts such as root locus, phase margin, bandwidth, etc., because he can relate these parameters to a good design. To use the modern approach, the designer must translate the design specifications into a performance index which he then proceeds to minimize or maximize to obtain the control that yields these specifications. The correlation between the performance index and performance criteria is the main subject of this chapter. In order to study the problem, three subobjectives are considered.

(1) To obtain the optimal control in terms of the system and performance index parameters for a simple system. The motivation behind this objective is the desire for a better understanding of the effect of these parameters on the resulting optimal system. This type of information is helpful in translating specifications into a performance index.

(2) To study the evaluated cost function in order to obtain its meaning in terms of a classical control problem.

(3) To investigate the optimal design techniques that are used to realize a desired system.

The infinite linear regulator problem is considered because it has a closedform mathematical solution and it is similar to the servomechanism problem.

2.2. Derivation of the Optimal Control. The purpose of this section is to derive the optimal control for the linear regulator of Figure 2-1 with the following performance index:

$$J = \int_0^{\infty} [\underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u}] dt \quad (2-1)$$

Q is a positive semidefinite state weighting matrix and P is a positive definite control weighting matrix. The integrand of Equation (2-1) is positive definite to ensure stability of the optimal system. The plant to be optimized is described by

$$\dot{\underline{x}} = A \underline{x} + B \underline{u} \quad (2-2)$$

The above performance measure expresses the desire to drive the state vector from an initial condition to the origin with a trade off between the system error and the amount of control energy expended. This desire is readily seen by making use of the end result. Minimization of this performance index leads to time invariant linear state variable feedback as the optimal control. For a second order linear time invariant plant the optimal control is

$$\underline{u} = -K_1 \underline{x}_1 - K_2 \underline{x}_2 \quad (2-3)$$

Substituting this result into Equation (2-1) and expanding yields

$$J = \int_0^{\infty} [q_{11} x_1^2 + q_{22} x_2^2] dt + \int_0^{\infty} p_{11} [K_1^2 x_1^2 + 2K_1 K_2 x_1 x_2 + K_2^2 x_2^2] dt \quad (2-4)$$

where

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \quad (2-5)$$

$$P = p_{11} \quad (2-6)$$

The first integral of Equation (2-4) represents the system error and the second integral represents control energy. The choice of  $q_{11}$ ,  $q_{22}$ , and  $p_{11}$  represents the emphasis on minimizing the system error as opposed to control energy.

The control interval is infinite and thus time is of no importance. For a finite upper limit on the performance index, the optimal control is time varying state feedback. In the optimal control literature it is shown that the optimal control is

$$u(x, t) = -P^{-1} B^T R(t) x \quad (2-7)$$

where  $R(t)$  is the solution to the matrix Ricatti equation.

$$\dot{R}(t) + Q - R(t) B P^{-1} B^T R(t) + R(t) A + A^T R(t) = 0 \quad (2-8)$$

Since the performance index has an infinite upper limit and the optimal system is assumed to be stable,  $\dot{R}(t)$  vanishes and  $R(t)$  becomes a time invariant matrix. This leads to the reduced Ricatti equation.

$$A^T R + R A - R B P^{-1} B^T R + Q = 0 \quad (2-9)$$

The optimal control, Equation (2-7), then reduces to

$$u(x) = -P^{-1}B^T R x \quad (2-10)$$

For the plant of Figure 2-1, the matrices of the reduced Riccati equation are

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \quad (2-11)$$

$$P = p_{11} \quad (2-12)$$

$$Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} \quad (2-13)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} \quad (2-14)$$

$$B = \begin{bmatrix} 0 \\ G \end{bmatrix} \quad (2-15)$$

R is a symmetric positive definite matrix. Substituting the above matrices into the reduced Riccati equation yields

$$\begin{bmatrix} \left(-\frac{G^2}{p_{11}} r_{12}^2 + q_{11}\right) & \left(r_{11} - a r_{12} - \frac{G^2}{p_{11}} r_{12} r_{22}\right) \\ \left(r_{11} - a r_{12} - \frac{G^2}{p_{11}} r_{12} r_{22}\right) & \left(2 r_{12} - 2 a r_{12} - \frac{G^2}{p_{11}} r_{22}^2 + q_{22}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2-16)$$

Application of Sylvester's Theorem<sup>14</sup> to obtain the requirements for a matrix to be positive definite yields the following solution to the Ricatti equation:

$$R_{11} = \frac{\sqrt{P_{11}g''}}{G} \sqrt{a^2 + G^2 \frac{g_{22}''}{p''} + 2G \sqrt{g''/p''}} \quad (2-17)$$

$$R_{12} = \frac{1}{G} \sqrt{g''p''} \quad (2-18)$$

$$R_{22} = \frac{P_{11}}{G^2} \left[ -a + \sqrt{a^2 + G^2 \frac{g_{22}''}{p''} + 2G \sqrt{g''/p''}} \right] \quad (2-19)$$

The optimal control is

$$u(\underline{x}) = -RBP^{-1}\underline{x} \quad (2-20)$$

$$u(\underline{x}) = - \begin{bmatrix} \frac{G}{p''} R_{12} \\ \frac{G}{p''} R_{22} \end{bmatrix}^T \underline{x} \quad (2-21)$$

$$u(\underline{x}) = - \begin{bmatrix} \sqrt{g''/p''} \\ \frac{1}{G} \left[ -a + \sqrt{a^2 + G^2 \frac{g_{22}''}{p''} + 2G \sqrt{g''/p''}} \right] \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2-22)$$

This result leads to the system in Figure 2-2 where

$$K_1 = \sqrt{g''/p''} \quad (2-23)$$

$$K_2 = \frac{1}{G} \left[ -a + \sqrt{a^2 + G^2 \frac{g_{22}''}{p''} + 2G \sqrt{g''/p''}} \right] \quad (2-24)$$

Equations (2-23) and (2-24) represent optimal feedback gains in terms of the plant parameters and the weighting factors of the performance index; but before discussing them, another approach is employed to check the results and show the effect of the parameters on the state trajectories.

A check on these results is obtained by using the Euler equation. However, this approach is valid only if the plant can be described in Normal (phase variable) Form which implies that the plant must be controllable.<sup>3</sup> For the plant in Figure 2-1

$$\dot{X}_1 = X_2 \quad (2-25)$$

$$\dot{X}_2 = -aX_2 + Gu \quad (2-26)$$

or

$$\ddot{X}_1 = -aX_2 + Gu \quad (2-27)$$

The performance index, Equation (2-1), can be written as

$$J = \int_0^{\infty} [g_{11}X_1^2 + g_{22}X_2^2 + u^2 p_{11}] dt \quad (2-28)$$

Solving Equation (2-26) for  $u$  and substituting the result into Equation (2-24) yields

$$J = \int_0^{\infty} \left[ g_{11}X_1^2 + g_{22}\dot{X}_1^2 + \frac{p_{11}}{G^2}\ddot{X}_1^2 + \frac{2ap_{11}}{G^2}\dot{X}_1\ddot{X}_1 + \frac{a^2p_{11}}{G^2}X_1^2 \right] dt \quad (2-29)$$

or

$$J = \int_0^{\infty} g(x_1, \dot{x}_1, \ddot{x}_1) dt \quad (2-30)$$

Substituting the integrand above into the following Euler equation

$$\frac{\partial g(x_1, \dot{x}_1, \ddot{x}_1)}{\partial x_1} - \frac{d}{dt} \left[ \frac{\partial g(x_1, \dot{x}_1, \ddot{x}_1)}{\partial \dot{x}_1} \right] + \frac{d^2}{dt^2} \left[ \frac{\partial g(x_1, \dot{x}_1, \ddot{x}_1)}{\partial \ddot{x}_1} \right] = 0 \quad (2-31)$$

yields

$$\ddot{X}_1 - \left[ G^2 \frac{g_{22}}{p_{11}} + a^2 \right] \ddot{X}_1 + G^2 \frac{g_{11}}{p_{11}} X_1 = 0 \quad (2-32)$$

Taking the Laplace transform of Equation (2-32) yields

$$s^4 - \left[ G^2 \frac{g_{22}}{p_{11}} + a^2 \right] s^2 + G^2 \frac{g_{11}}{p_{11}} = 0 \quad (2-33)$$

Equation (2-33) can be factored into the following:

$$\left[ s + \sqrt{A - \sqrt{A^2 - B}} \right] \left[ s - \sqrt{A - \sqrt{A^2 - B}} \right] \left[ s + \sqrt{A + \sqrt{A^2 - B}} \right] \left[ s - \sqrt{A + \sqrt{A^2 - B}} \right] = 0 \quad (2-34)$$

where

$$A = G^2 \frac{g_{22}}{2 p_{11}} + \frac{a^2}{2} \quad (2-35)$$

$$B = G^2 \frac{g_{11}}{p_{11}} \quad (2-36)$$

Two of the roots of Equation (2-34) are in the right half plane and thus must be eliminated to satisfy the boundary condition

$$\underline{X}(\infty) = 0 \quad (2-37)$$

Thus, Equation (2-34) reduces to

$$\left[ s + \sqrt{A - \sqrt{A^2 - B}} \right] \left[ s + \sqrt{A + \sqrt{A^2 - B}} \right] = 0 \quad (2-38)$$

Taking the inverse Laplace transform yields

$$\ddot{X}_1 + \dot{X}_1 \left[ \sqrt{A - \sqrt{A^2 - B}} + \sqrt{A + \sqrt{A^2 - B}} \right] + X_1 \sqrt{B} = 0 \quad (2-39)$$

Equation (2-39) represents the optimal state trajectory for the system based on the chosen performance index. But, from Equation (2-27)

$$\ddot{X}_1 = -a\dot{X}_1 + Gu \quad (2-40)$$

Substituting this equation into Equation (2-39) and solving for  $u$  yields

$$u = -\frac{\sqrt{B}}{G} X_1 - \frac{1}{G} \left[ -a + \sqrt{A - \sqrt{A^2 - B}} + \sqrt{A + \sqrt{A^2 - B}} \right] X_2 \quad (2-41)$$

Therefore, the optimal feedback gains are

$$K_1 = \frac{\sqrt{B}}{G} \quad (2-42)$$

$$K_2 = \frac{1}{G} \left[ -a + \sqrt{A - \sqrt{A^2 - B}} + \sqrt{A + \sqrt{A^2 - B}} \right] \quad (2-43)$$

Equation (2-43) is simplified as follows:

$$(K_2 G + a)^2 = \sqrt{2A + 2\sqrt{B}} \quad (2-43a)$$

Solving for  $K_2$  yields

$$K_2 = \frac{1}{G} \left[ -a + \sqrt{2A + 2\sqrt{B}} \right] \quad (2-44)$$

With the appropriate substitutions for A and B, the optimal feedback gains are

$$K_1 = \sqrt{\frac{g_{11}}{p_{11}}} \quad (2-45)$$

$$K_2 = \frac{1}{G} \left[ -a + \sqrt{a^2 + G^2 \frac{g_{22}}{p_{11}} + 2G \sqrt{\frac{g_{11}}{p_{11}}}} \right] \quad (2-46)$$

These results are identical to those found by solving the reduced Ricatti equation.

The following results are stated without proof for a first order plant shown in Figure 2-3:

$$K_1 = \frac{1}{G} \left[ -a + \sqrt{G^2 \frac{g_{11}}{p_{11}} + a^2} \right] \quad (2-47)$$

The above approach not only serves as a check but also demonstrates the effect of the plant and cost index parameters on the optimal trajectories. To obtain general results such as these for higher order systems brings about mathematical difficulties. The digital computer must be employed to obtain specific results for specific systems. The solution to the

reduced Ricatti equation is obtained by numerically integrating the matrix Ricatti differential equation backwards in time with the following boundary condition:

$$R(t = 0) = 0 \quad (2-48)$$

For large negative time the  $R$  terms will approach constant values that are the solution to the reduced Ricatti equation. An example of this procedure is given in section 2.3. However, the general results obtained for the second order system serve to illustrate the effect of the plant and performance measure parameters on the optimal system.

With attention focused on Equations (2-45) and (2-46), some general results are obtained. If the elements of the  $Q$  matrix are held constant and  $p_{11}$  made large, the feedback gains are small. This result is reasonable since large weight on control energy relative to the weights on the system error expresses the desire to use small amounts of control energy at the expense of large states, i.e., system error. For a constant  $p_{11}$  and for large values for the elements of the  $Q$  matrix, the feedback gains are large. Again this result is reasonable since the emphasis is now shifted to small system error at the expense of large controls, i.e., large feedback gains. However, the  $Q$  and  $P$  matrices actually have the same effect on the feedback gains since the optimal control is a linear combination of the states. That is, gains are made large by increasing  $Q$  or decreasing  $p_{11}$ . The same optimal system results from an infinite number of choices of  $Q$  and  $p_{11}$  since the optimal control is a function only of the ratios  $q_{11}/p_{11}$  and  $q_{22}/p_{11}$ . Thus, unless each state is of separate concern, the  $Q$  matrix should be the identity matrix and the  $p_{11}$  term used to achieve

the desired optimal system. This result will be utilized in the section on modeling. The pole and gain of the plant also affect the feedback gains. For large pole values the size of  $K_2$  decreases. This is expected because large pole values imply a small time constant for the open loop plant. Thus, the plant is able to decay to the origin with little assistance from the control.

The characteristic equation describing the system in Figure 2-2 is

$$\ddot{X}_1 + (a + K_2 G) \dot{X}_1 + K_1 G X_1 = 0 \quad (2-49)$$

Therefore

$$\omega_n = \sqrt{K_1 G} = \sqrt{G \sqrt{\frac{q_{11}}{p_{11}}}} \quad (2-50)$$

$$\xi = \frac{a + K_2 G}{2 \omega_n} = \frac{1}{2 \sqrt{G \sqrt{q_{11}/p_{11}}}} \sqrt{a^2 + G^2 \frac{q_{22}}{p_{11}} + 2G \sqrt{\frac{q_{11}}{p_{11}}}} \quad (2-51)$$

where  $\omega_n$  is the undamped natural frequency and  $\xi$  is the damping ratio. From Equations (2-46) and (2-47), large  $Q$  matrices relative to  $p_{11}$  produce highly damped systems with large bandwidths. But note that  $\xi$  can exceed unity. One would expect that some sort of limit on  $\xi$  would result from the optimization process. For large values of  $p_{11}$  relative to the terms of the  $Q$  matrix, optimal systems which are highly oscillatory with small bandwidths are produced. But, there is a compromise between these two extremes which is obtained by trade-offs between  $q_{11}$ ,  $q_{22}$ , and  $p_{11}$ . Thus, even though the optimal control has been found, the classical ideas on what constitutes a good design must be introduced to express the ultimate objective. The performance

index chosen yields a class of optimal systems and any specific one of the class is chosen by selecting the appropriate Q and P matrices. However, the general results obtained for the second order system do give some feeling on how to select the parameters of the performance index. For higher order systems the exact relationships between the performance index parameters and the classical parameters are not known, and thus some other approach must be taken. This new approach is modeling.

2.3 Modeling. As noted in the previous section it is difficult to translate the classical design criteria into the performance index parameters for systems higher than second order. Even the procedure of adjusting the Q and P matrices until an acceptable time constant and percent overshoot are obtained gives no information about the bandwidth, phase margin, etc. However, this problem can be eliminated by using a procedure similar to one employed by the classical approach. In the classical approach, one often adds components or feedback loops so that the new system behaves like a known differential equation. This model differential equation represents a system that has the desired classical specifications. If this model is incorporated into the performance index so that the mathematical process of minimization is applied to the difference between the system response and the model response, then the optimization theory takes on meaning and practical significance. Schultz and Melsa<sup>14</sup> present such a modeling approach to the design of a linear regulator.

Consider a plant described in Normal Form. Any quadratic index of the form

$$J = \int_0^{\infty} [\underline{x}^T Q \underline{x} + p_{11} u^2] dt \quad (2-52)$$

where the states are phase variables, is reducible to

$$J = \int_0^{\infty} \left[ (L^T \underline{x})^2 + \frac{d}{dt} (\underline{x}^T S \underline{x}) + p_{11} u^2 \right] dt \quad (2-53)$$

S is a symmetric constant matrix of order one less than the system, and L is a column vector of system order. That is, the first term of the quadratic form is reducible to a perfect square and an exact differential. Again, it must be emphasized that this result is valid only when the states are phase variables. The exact differential is not affected by the minimization process because

$$\int_0^{\infty} \frac{d}{dt} (\underline{x}^T S \underline{x}) dt = \left. \underline{x}^T(t) S \underline{x}(t) \right|_{t=0}^{t=\infty} = -\underline{x}^T(0) S \underline{x}(0) \quad (2-54)$$

From Equation (2-54) it is evident that the exact differential term represents a fixed value of cost that is determined only by the initial conditions. The term is positive because the terms of the S matrix are always negative. The derivation of the L and S matrices for second and third order systems is given in Appendix B. The general recursion formula for higher order systems may be obtained from Reference 14. The performance index, for the Normal Form representation, is equivalent to

$$J = \int_0^{\infty} \left[ (L^T \underline{x})^2 + p_{11} u^2 \right] dt \quad (2-55)$$

With this performance index, the L matrix has to be selected instead of a Q matrix. But this form allows one to incorporate the desired model.

Consider, for the moment, that  $p_{11}$  is zero. Then the cost function is zero if

$$\underline{L}^T \underline{\dot{X}} = l_1 \dot{x}_1(t) + l_2 \ddot{x}_1(t) + l_3 \dddot{x}_1(t) + \dots + l_n \dot{x}_1^{(n-1)}(t) = 0 \quad (2-56)$$

for all time. By the proper choice of the L matrix, Equation (2-56) will be the model differential equation. The Q matrix, which is needed to solve the Ricatti equation, is derived from the L matrix using the formulas of Appendix B. Then,  $p_{11}$  is decreased until the optimal system response is close to the model response. However, the system response cannot be identical to the model response because as  $p_{11}$  approaches zero the feedback gains become extremely large. Stability problems are also encountered due to the fact that  $p_{11}$  is then semidefinite. The value of the cost function then represents how close, on an integral square error basis, the system response is to the model. Thus, one is prevented from attaining the desired model response by physical restrictions on the size of the feedback gains. Note that the model equation must be at least one order less than the plant in order that the L matrix not exceed the system order. This presents no problem since one usually designs higher order systems to behave like second order systems. This modeling technique is also applicable to plants which have zeros. In summary, the procedure of this technique is to select the L matrix from the coefficients of the differential equation representing the desired response and then determining the elements of the Q matrix from the relations in Appendix B. With this Q matrix, the Ricatti equation is solved with various values of  $p_{11}$  until the maximum permissible feedback gains are obtained. The following examples demonstrate this procedure.

Example 2.1 Consider the plant of Figure 2-1 where

$$G = 1 ; a = 2 \quad (2-57)$$

and the desired model is

$$\dot{X}_1 + 2X_1 = 0 \quad (2-58)$$

Since the L matrix comes from the model differential equation, it follows from Equations (2-56) and (2-58) that

$$L = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (2-59)$$

Using this L matrix and the relations of Appendix B, the Q matrix is determined to be

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \quad (2-60)$$

With this Q matrix, the feedback gains are determined from the relations for the second order system of the previous section. Figure 2-4 shows the desired model response as well as the optimal system responses for various values of  $p_{11}$ . Note that as  $p_{11}$  is decreased the optimal system response approaches the model response. Figure 2-14 shows the trade-off between the size of the optimal gains and the cost (proximity of optimal response to model response). The next example is a more practical one since the plant is third order and the desired model is second order.

Example 2.2 Consider the plant of Figure 2-5 where

$$G = 1 ; a = 3 ; b = 5 \quad (2-61)$$

and the desired model is

$$\ddot{X}_1 + 2\dot{X}_1 + 2X_1 = 0 \quad (2-62)$$

Using Equations (2-56) and (2-62), the L matrix is

$$L = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad (2-63)$$

From the results of Appendix B for a third order plant, the Q matrix is determined to be

$$Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2-64)$$

Since there are no closed form solutions to the reduced Ricatti equation, the matrix Ricatti differential equation must be numerically integrated to obtain the solution to the reduced Ricatti equation. This solution is obtained by integrating the matrix Ricatti differential equation backward in time with initial conditions

$$R(t = 0) = \underline{0} \quad (2-65)$$

For sufficiently large negative time, the terms of the R matrix will approach constant values and these values are the solution to the reduced Ricatti equation. The optimal feedback gains are then obtained from

$$K = R B P^{-1} \quad (2-66)$$

The solutions to the matrix Ricatti differential equation for various values of  $p_{11}$  are shown in Figures (2-7), (2-8), (2-9), and (2-10). The system responses for the feedback gains associated with the various  $p_{11}$  are shown in Figure (2-11). Again, note that as  $p_{11}$  decreases the optimal system response approaches the model response. Figure 2-15 shows the trade-off between the size of the optimal gains and the cost (closeness of the optimal response to the model response). The

mathematical formulation of the first order differential equations comprising the matrix Ricatti differential equation is given in Appendix C. This modeling technique provides more information about the meaning of optimality with respect to a quadratic index than the techniques to be discussed next. Since the  $Q$  matrix and the  $p_{11}$  term affect the optimal control in the same way, it is simpler to fix the  $Q$  matrix and vary the  $p_{11}$  term to obtain the desired optimal system. Once the  $Q$  matrix is chosen, optimality is also defined since there is a unique  $L$  matrix associated with the chosen  $Q$ .

Another modeling technique is proposed by Tyler<sup>9</sup>. This technique also uses the quadratic performance index but defines a new set of state variables that are the difference between the model states and the states of the plant. This index is then minimized. This procedure also leads to the matrix Ricatti differential equation. This procedure, unlike the previous one, requires the adjustment of both the  $Q$  and  $P$  matrices to yield a satisfactory system response. These two modeling techniques bring to light several conclusions about the optimal control approach to the design of regulators and servomechanisms. The optimization technique is a mathematical method of introducing a set of variable parameters into the actual system that provide the designer with a more useful tool with which to cope with system constraints such as gains. However, the ultimate objective is still expressed in the form of a differential equation. The optimal controls approach is a search for the performance index that yields the desired differential equation. The physical significance of the value of the cost index is also questionable and, therefore, is considered next.

2.4 The Evaluated Cost Index. Consider again the performance index of the linear regulator. Kalman<sup>3</sup> shows that the value of this index for a stable system is

$$J_{op} = \underline{X}^T R \underline{X} \quad (2-67)$$

where R is the solution to the reduced Ricatti equation. For the second order optimal system of Figure 2-2, Equation (2-67) yields

$$J_{op} = p_{11} \frac{K_1}{G} (a + K_2 G) X_1^2(0) + 2 p_{11} \frac{K_1}{G} X_1(0) X_2(0) + p_{11} \frac{K_2}{G} X_2^2(0) \quad (2-68)$$

Section 2.1 stressed the fact that there are many values of Q and P that yield the same optimal system. This is also true for the value of the cost index. One could then argue that the value of the cost index has no significance. In general this is true. But, for the modeling technique of Schultz and Melsa, the cost does have meaning. For that technique, zero cost corresponded to a system response identical to the model response. Thus, the cost represents the deviation of the system response from the model response in an integral square error sense.

Even though the numerical value of the cost provides little information of interest in the design problem, it does provide some insight into the mathematical process of minimization. The cost function

$$J = \int_0^{\infty} (\underline{X}^T Q \underline{X} + \underline{u}^T P \underline{u}) dt \quad (2-69)$$

is evaluated for a non-optimal second order system by assuming solutions for  $X_1(t)$ ,  $X_2(t)$ , and  $u$  as follows:

$$X_1(t) = \alpha_1 e^{-\lambda_1 t} + \alpha_2 e^{-\lambda_2 t} \quad (2-70)$$

$$\dot{X}_2(t) = \dot{X}_1(t) = -\alpha_1 \lambda_1 e^{-\lambda_1 t} - \alpha_2 \lambda_2 e^{-\lambda_2 t} \quad (2-71)$$

$$u(t) = -K_1 X_1(t) - K_2 X_2(t) \quad (2-72)$$

The feedback gains of Equation (2-72) are not the optimal gains and thus are not the same as Equations (2-45) and (2-46). For the moment, consider them as arbitrary values. Substituting these solutions into Equation (2-69) and carrying out the integration yields

$$\begin{aligned} J = & \left[ (K_1^2 + \frac{g_{11}}{p_{11}}) + \frac{(K_1^2 + \frac{g_{11}}{p_{11}})(a + K_2 G)^2}{K_1 G} + K_1 G (K_2^2 + \frac{g_{22}}{p_{11}}) \right. \\ & \left. - 2 K_1 K_2 (a + K_2 G) \right] \frac{p_{11} X_1^2(0)}{2(a + K_2 G)} + \frac{p_{11} (K_1^2 + \frac{g_{11}}{p_{11}})}{K_1 G} X_1(0) X_2(0) \\ & + \left[ \frac{K_1^2 + \frac{g_{11}}{p_{11}}}{K_1 G} + K_2^2 + \frac{g_{22}}{p_{11}} \right] \frac{p_{11} X_2^2(0)}{2(a + K_2 G)} \end{aligned} \quad (2-73)$$

Equation (2-68) may be obtained from this relation by substituting the optimal gains. However, note that Jop, Equation (2-68), is a positive definite function, whereas J, Equation (2-73), may not be positive definite. The positive definite function, Jop, plots on the  $X_1$  versus  $X_2$  plane as a family of concentric ellipses as shown in Figure 2-12. The constant cost curves decrease continually toward the origin. However, Equation (2-73) will plot as a family of curves that may overlap since it is not necessarily a positive definite function. If the optimal feedback gains are used, the state trajectory on the  $X_1$  versus  $X_2$  plane will cross the family of constant cost curves in a manner such that the cost at any instant of time on the trajectory is less

than that for any previous instant of time. But, if non-optimal gains are used, this will not be true. The reason it is true for the positive definite function is due to a theorem by Kalman<sup>3</sup> which is discussed in Chapter III. The theorem states, under certain assumptions, that  $J_{op}$  is a Lyapunov function and that

$$\frac{d}{dt} J_{op} = - \left[ \underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u} \right] \quad (2-74)$$

is a negative definite function because of the previous assumption on  $P$  and  $Q$ . Therefore, the state trajectory must transit the  $J_{op}$  curves in the manner described above. One might suspect that the state trajectory on the  $X_1$  versus  $X_2$  plane would be normal to the  $J_{op}$  curves. This is not true because

$$\frac{d}{dt} J_{op} = \left[ \frac{\partial J_{op}}{\partial \underline{x}} \right]^T \frac{d \underline{x}}{dt} = \left[ \frac{\partial J_{op}}{\partial \underline{x}} \right]^T \dot{\underline{x}} \quad (2-75)$$

Thus, as seen by Equation (2-75), the time rate of crossing is not only a function of the gradient of the  $J_{op}$  curves but is also a function of the state equations. The shape and amount of axis rotation of these  $J_{op}$  curves depends on the choice of  $Q$  and  $P$ , as well as the plant parameters. Thus, when more weight is put on one of the states than the other, the effect is a rotation of the axes of the family of cost curves so as to restrict the motion of the state trajectory. Figure 2-13 is a plot of these cost curves along with the optimal trajectory for Example 2.1. The value of  $p_{11}$  for Figure 2-13 is unity.

A reasonable conclusion to be drawn from this chapter is that in the design of a regulator or servomechanism the optimal controls approach is superior to the classical approach in some respects and inferior in

other respects. The optimal controls approach does allow the engineer, inexperienced in the classical techniques, to design a control system to meet the required specifications. With the use of the modeling technique of Schultz and Melsa<sup>14</sup>, the modern approach gives the engineer a single number, the cost, that represents the proximity of the optimal system response to the model response (integral square error sense). The implementation of the optimal control is another problem. The finite linear regulator problem leads to an optimal control that is time varying state feedback. The servomechanism problem has an optimal control which is time varying state feedback plus a function of the desired reference states. The synthesis of these controls is difficult. Therefore, in most cases, the experience factor is merely shifted to the synthesis of the optimal control. The linear regulator was considered in this chapter because of the simple mathematics involved, and also because it offered a challenge to the meaning of optimality with respect to the design of a regulator to meet certain specifications. The quadratic performance index was discussed in detail. The effect of block diagram manipulations on this index is considered in Appendix D.

This chapter used the linear regulator to study the meaning of optimality. For this particular problem, the quadratic performance index with a finite upper limit led to a time varying system. When the optimal controls approach is used, the resulting optimal system is usually non-linear, time varying, or both. Stability then becomes a subject of primary concern. Lyapunov stability theory is the primary tool that is used to study the stability of these type systems. Like the optimal controls approach, Lyapunov stability theory is based on

the time domain description of systems. Chapter III will introduce some of the stability criteria that are useful in determining the stability of a class of time varying systems for which stability can be studied directly.

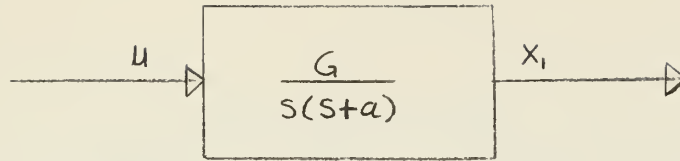


Figure 2-1. Second Order Plant

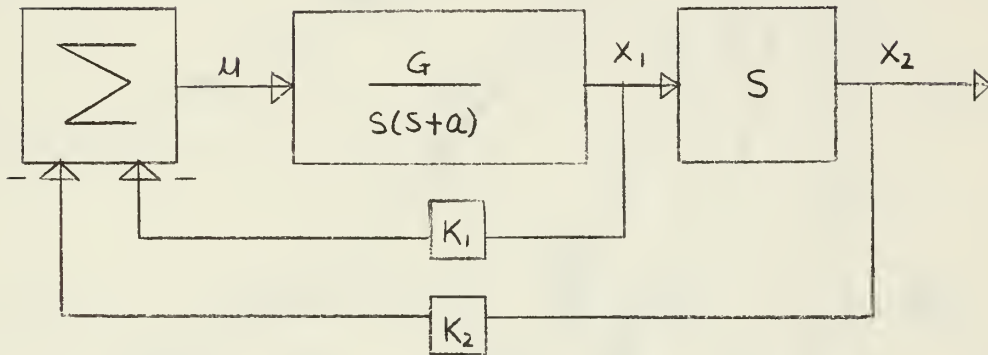


Figure 2-2. Optimal System

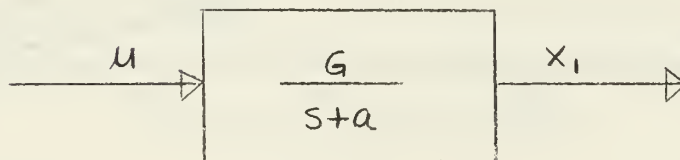


Figure 2-2. First Order Plant

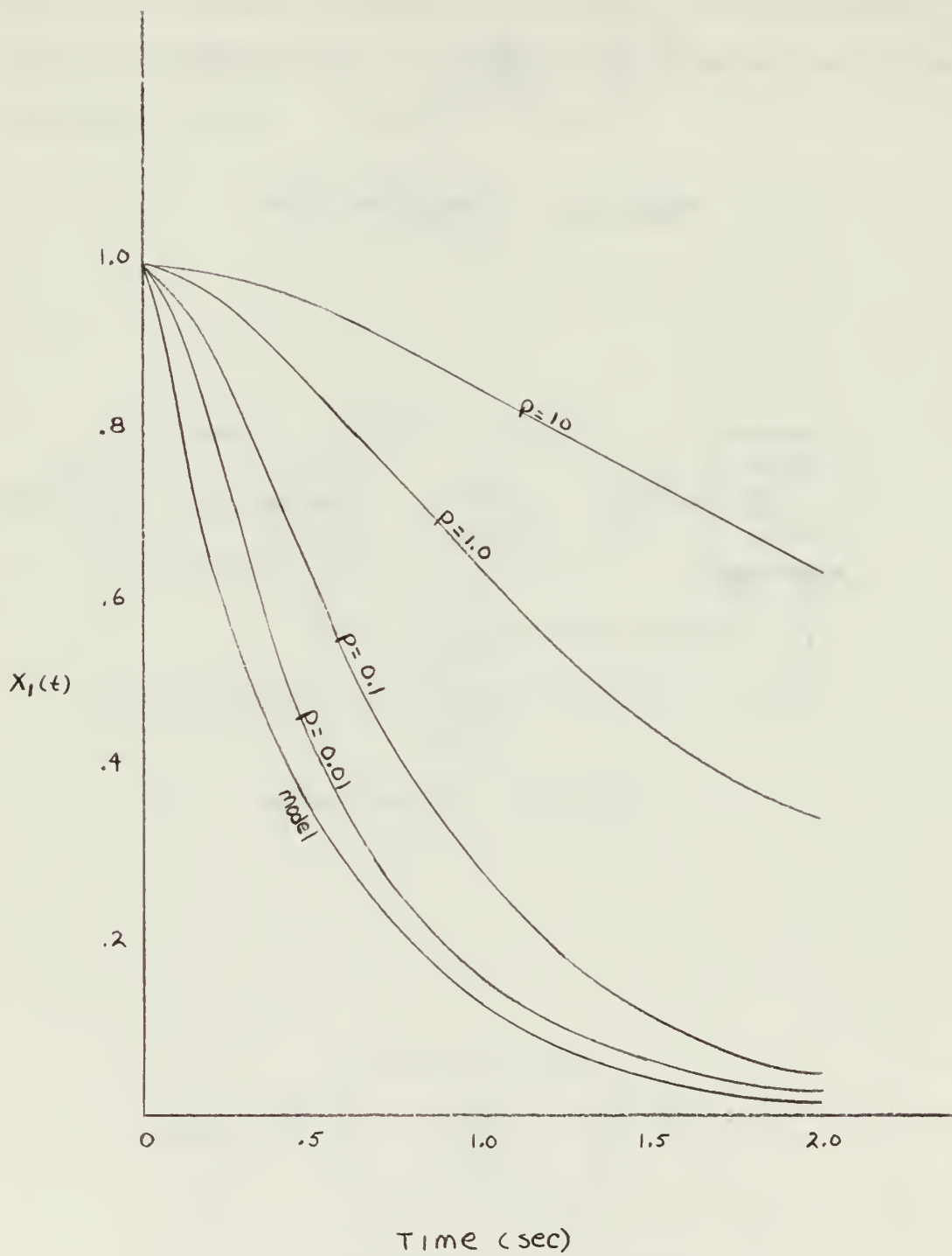


Figure 2-4. Optimal System Responses and Model Response

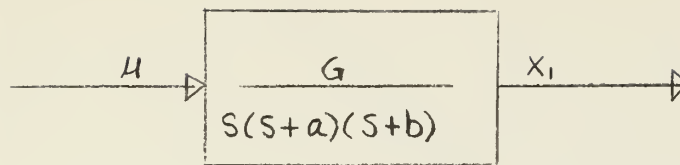


Figure 2-5. Third Order Plant

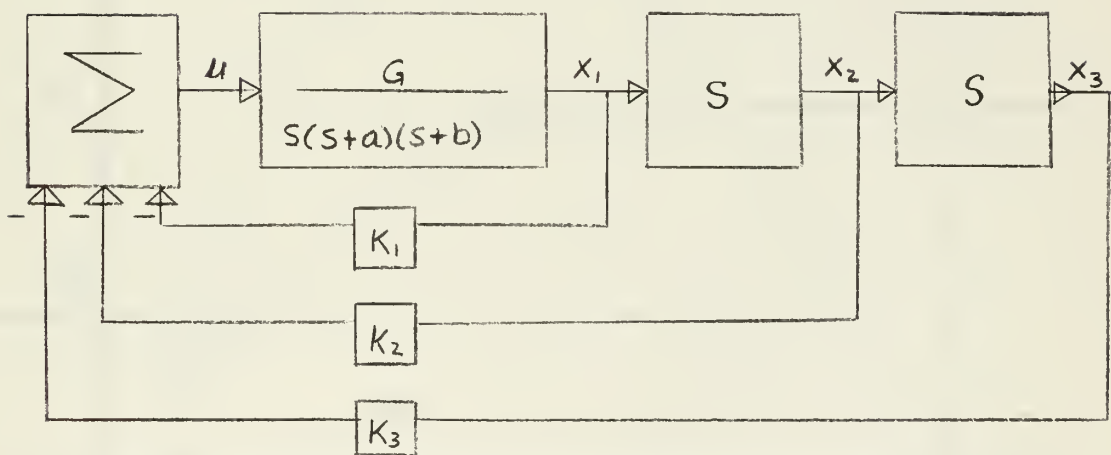


Figure 2-6. Optimal System

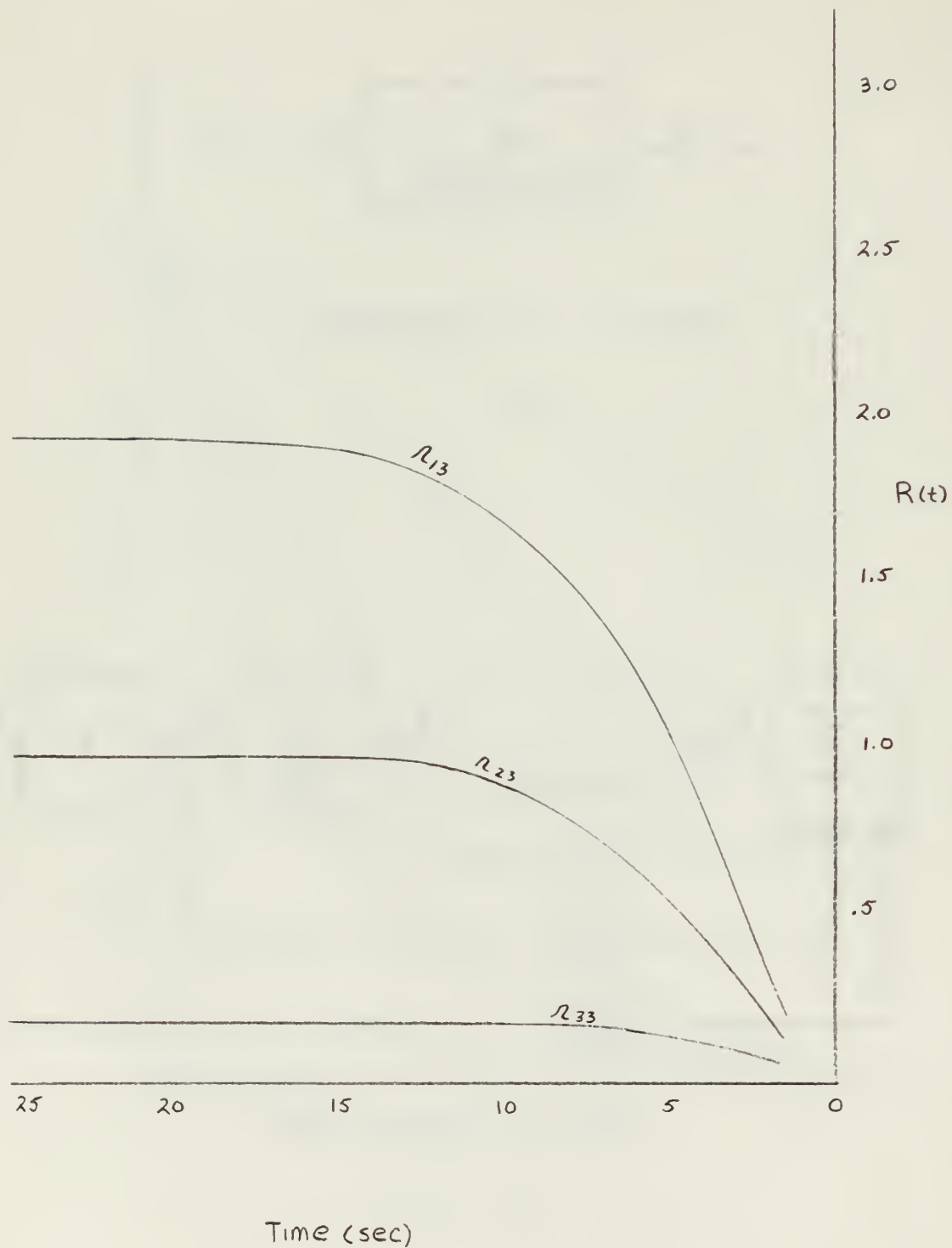


Figure 2-7. Steady State Solution to Ricatti Equation for  $p_{11} = 1.0$

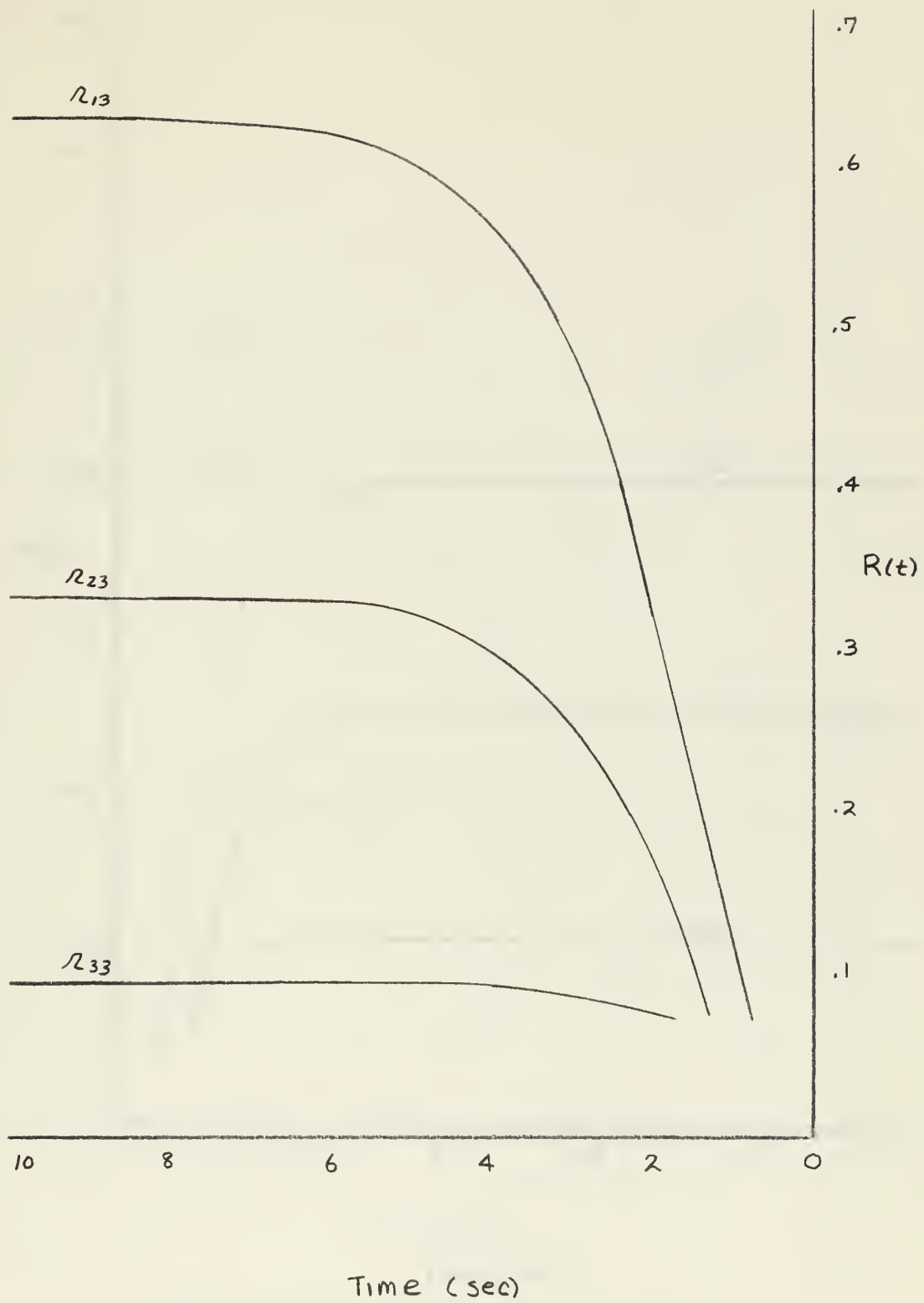


Figure 2-8. Steady State Solution to Riccati Equation for  $p_{11} = 0.1$

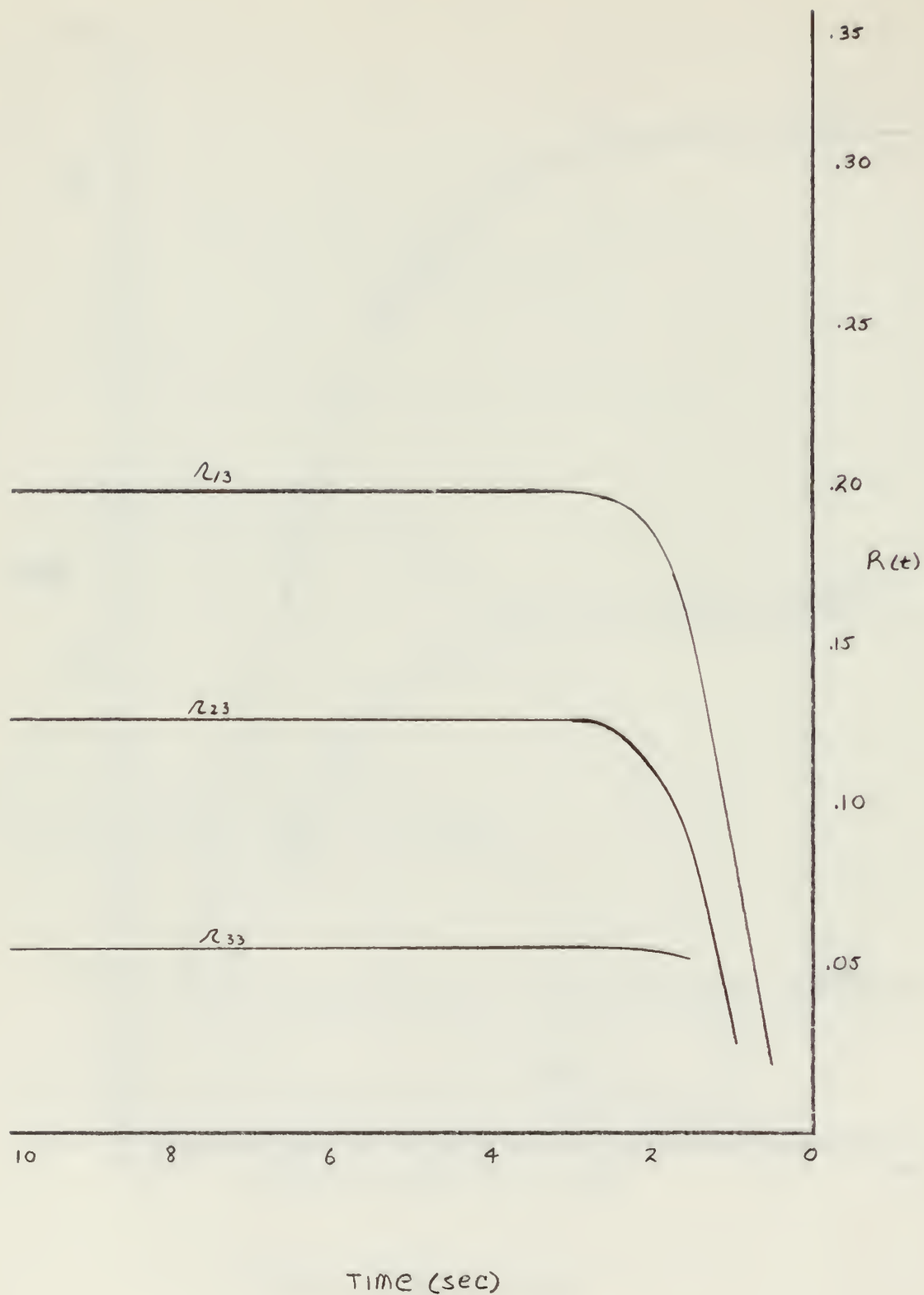


Figure 2-9. Steady State Solution to Riccati Equation for  $p_{11} = 0.01$

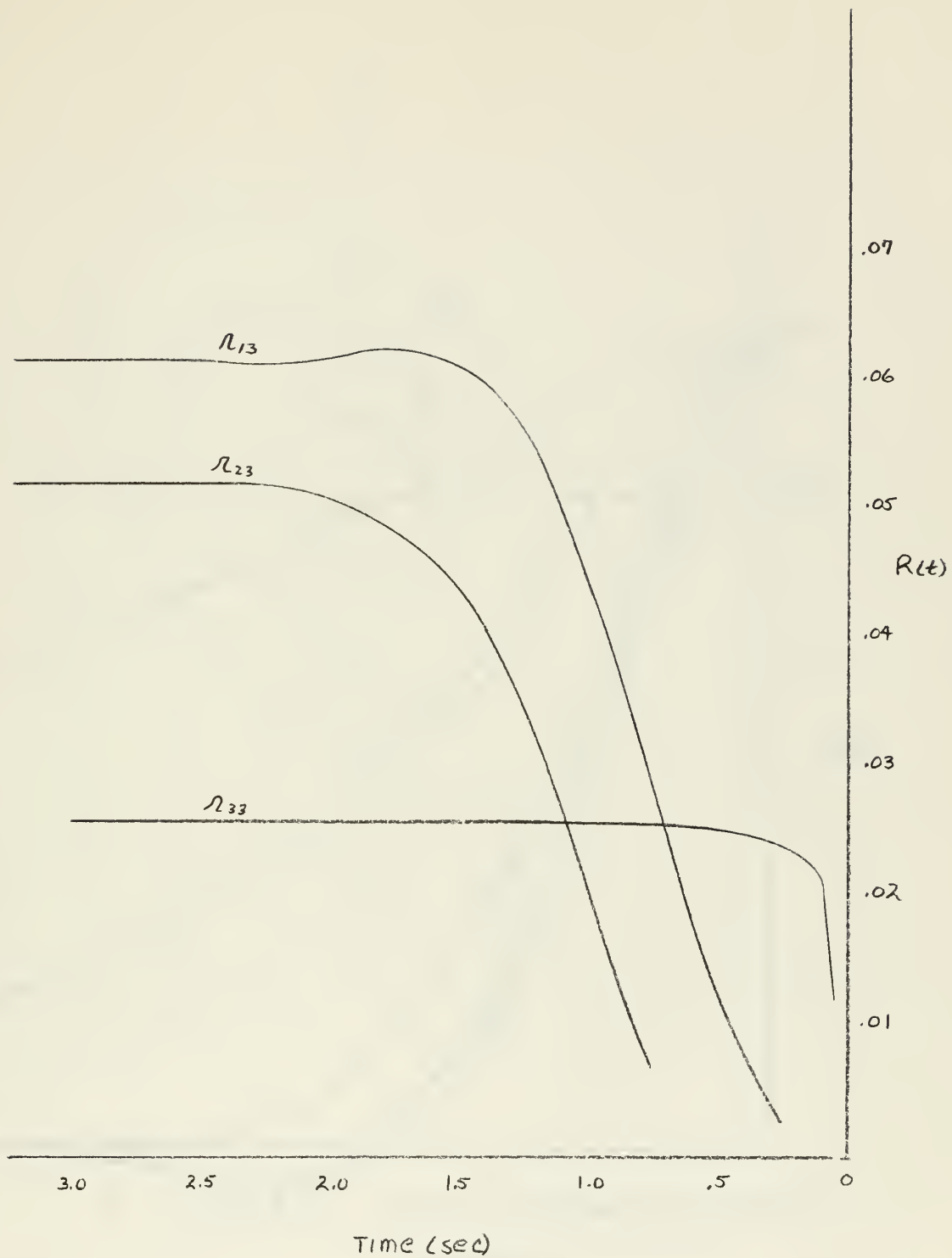


Figure 2-10. Steady State Solution to Ricatti Equation for  $p_{11} = 0.001$

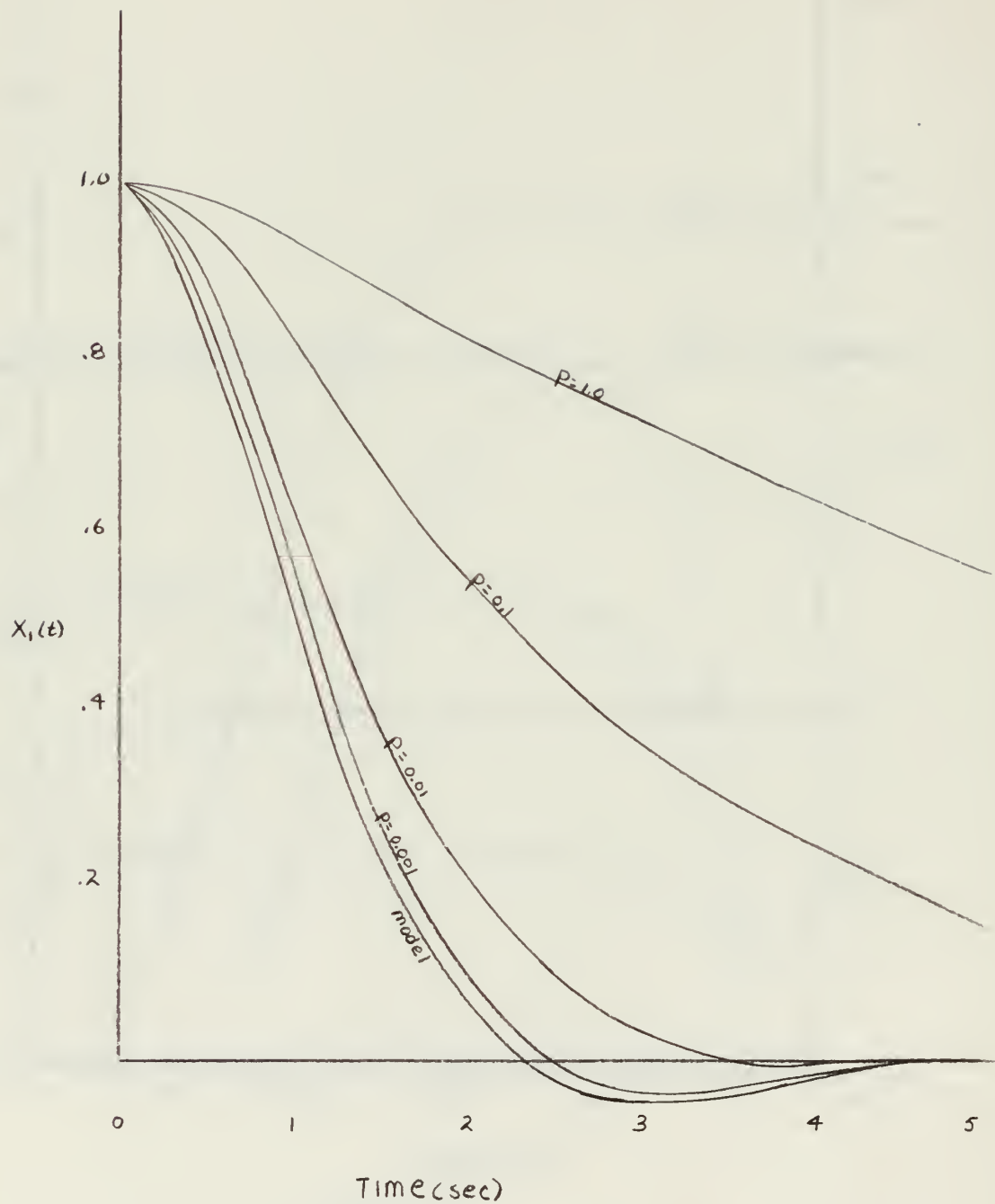


Figure 2-11. Optimal System Responses  
and Model Responses

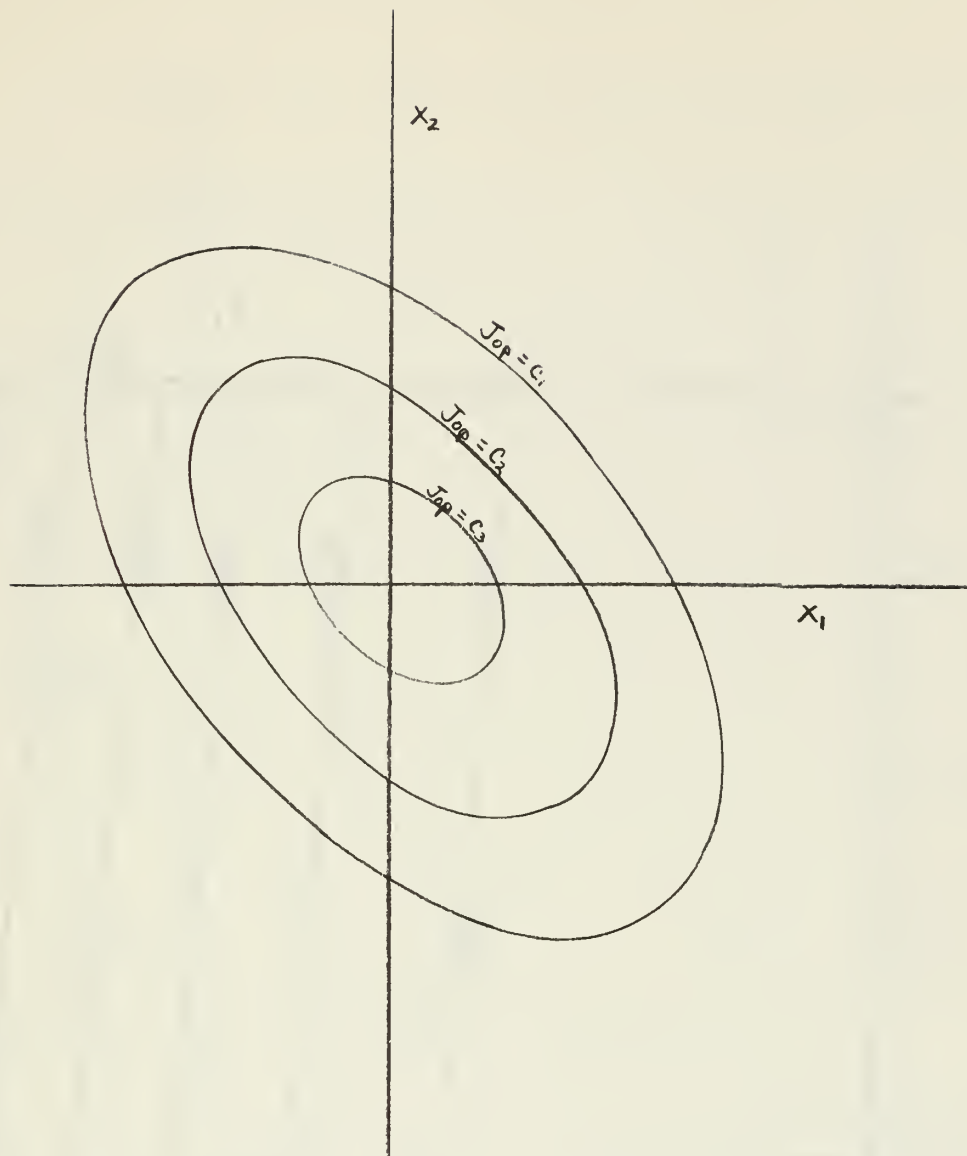


Figure 2-12. Curves of Constant Cost ( $C_3 > C_2 > C_1$ )

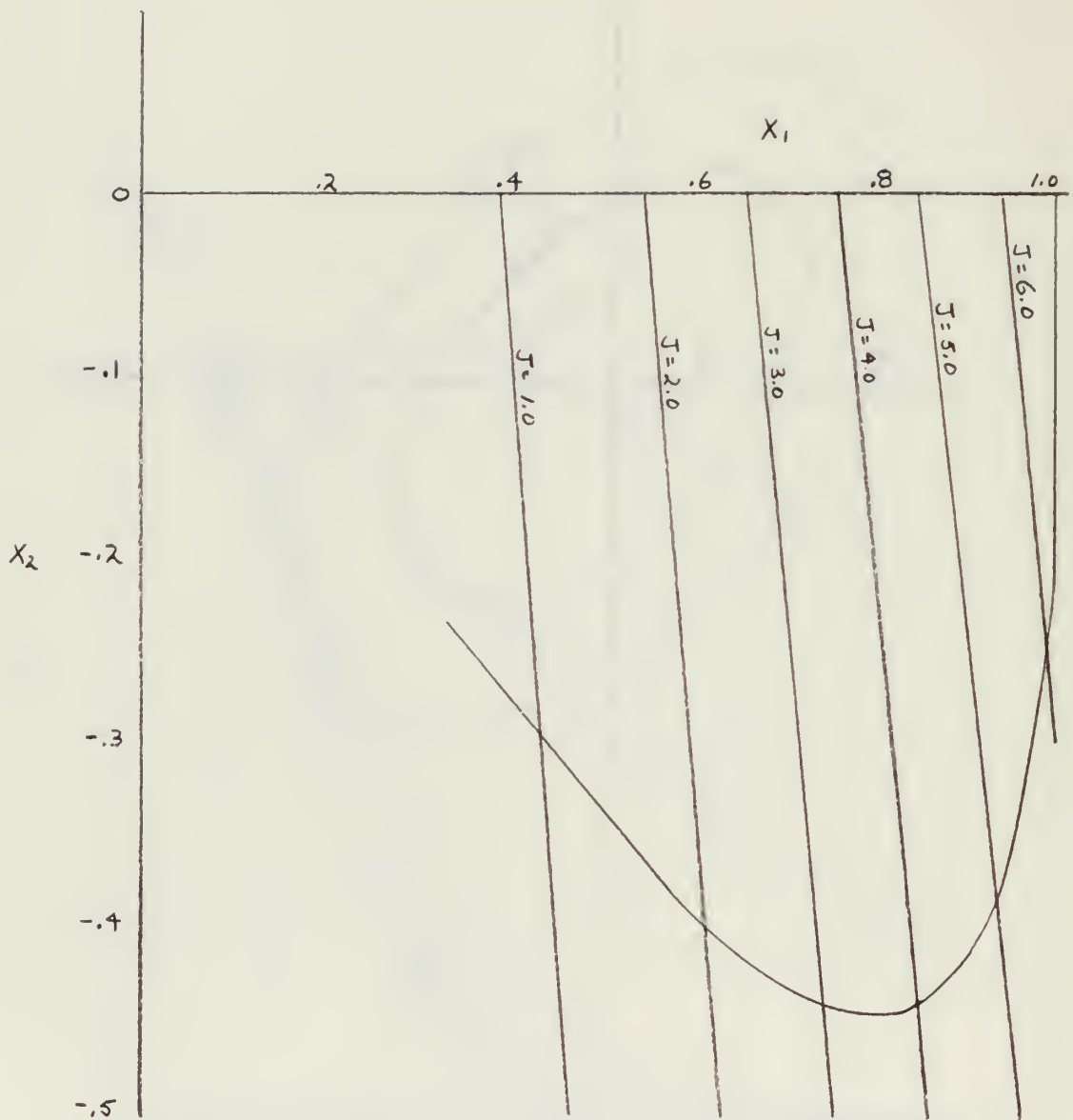


Figure 2-13. Optimal  $X_1$  vs  $X_2$  Trajectory  
with Constant Cost Curves

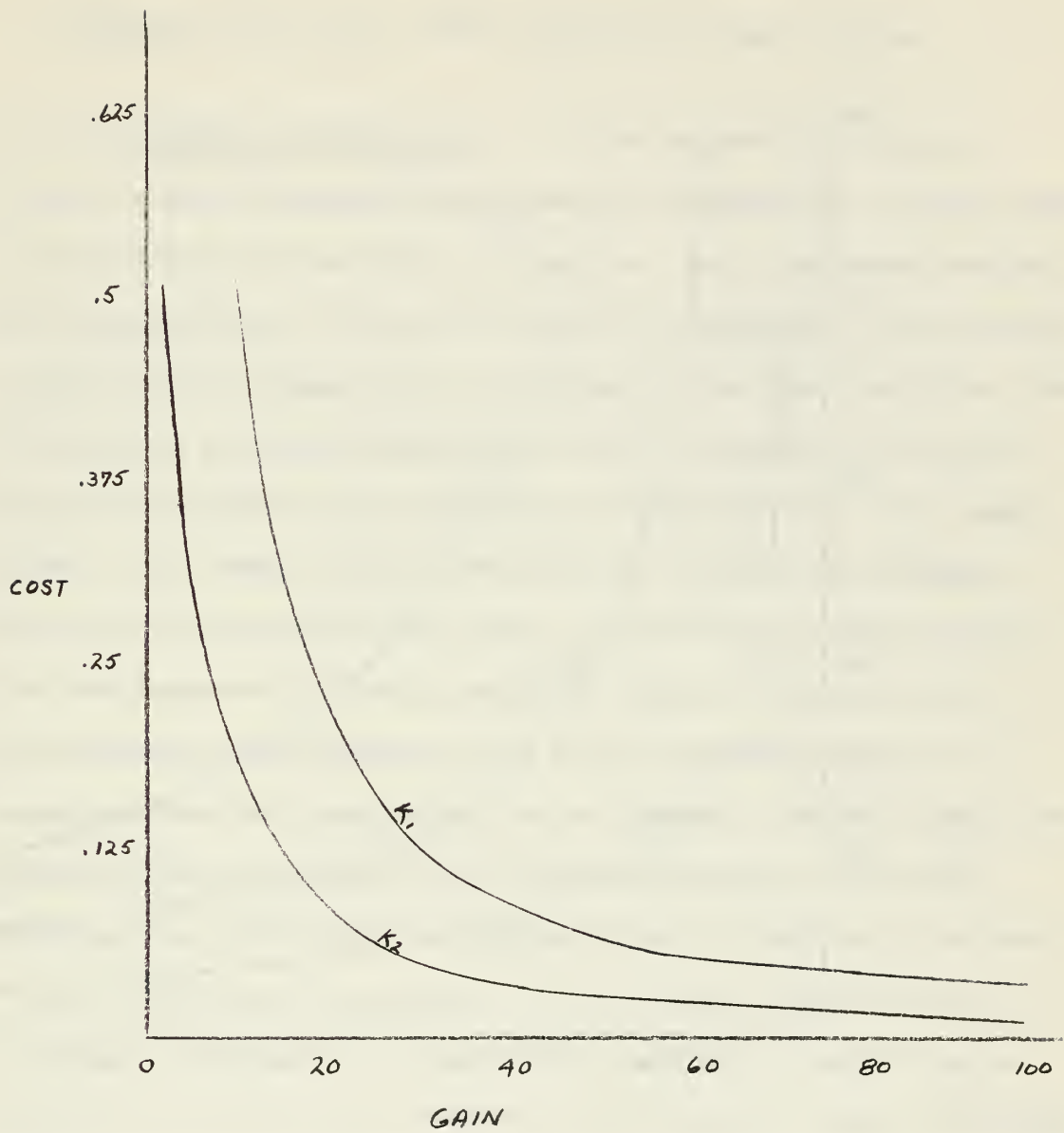


Figure 2-14. Cost Versus Optimal Feedback Gains for Example 2.1

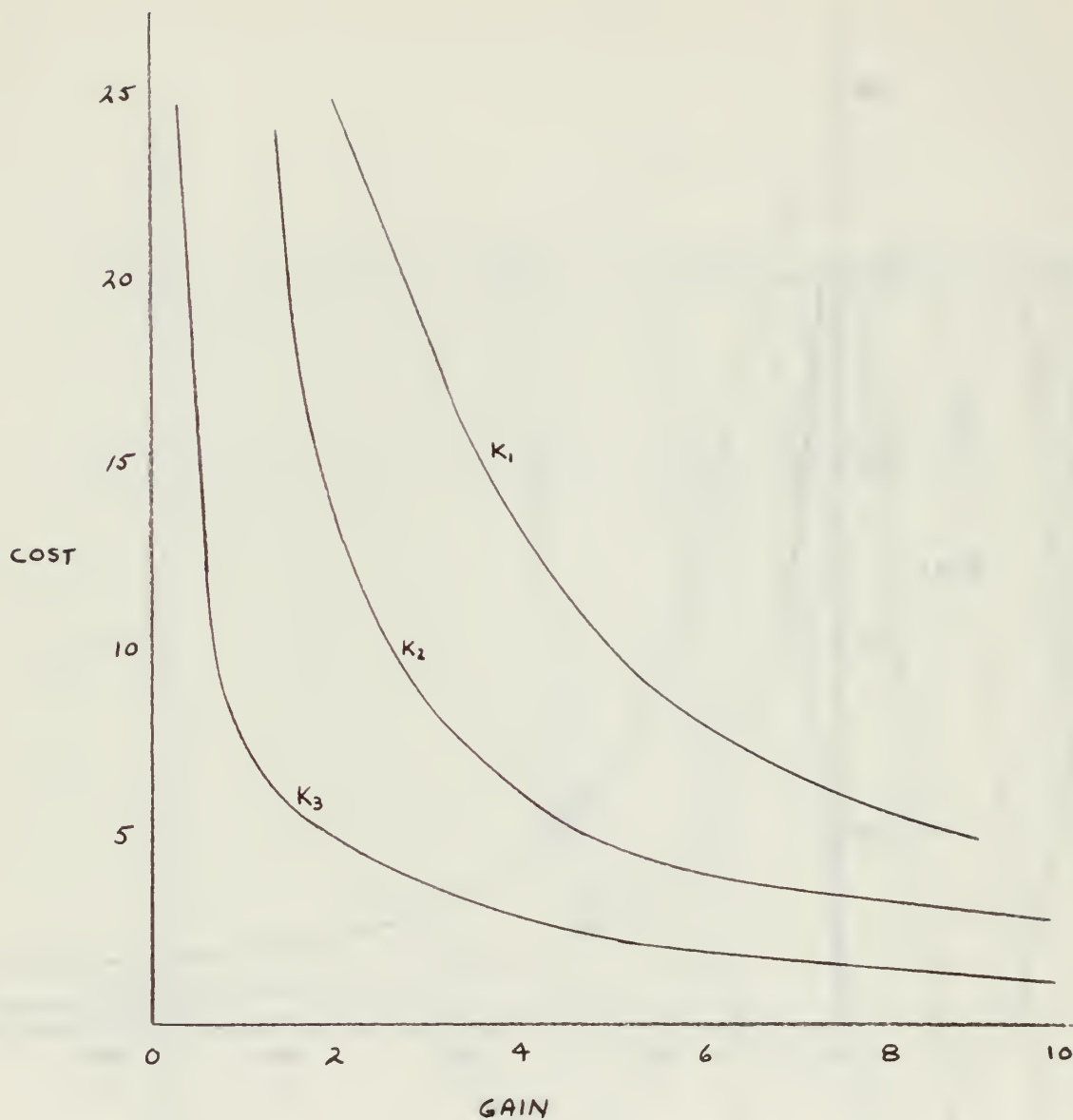


Figure 2-15. Cost Versus Optimal Feedback Gains for Example 2.2

## Stability of a Class of Time Varying and Optimal Systems

3.1. Lyapunov's Second Method. In the analysis and design of control systems an important question to be answered in the early stages is the question of stability. Not only is a yes or no answer desired, but also the limits of stability must be investigated. Various design criteria such as phase margin and the Routh criterion express the limits of stability graphically and algebraically, respectively. Thus, the range of values that system parameters can take on is of vital importance in the design problem. The study of stability can be broken down into two general areas. First, the stability of linear systems is straightforward due to the Routh and Nyquist criteria. Second, the stability of non-linear as well as time varying systems is not straightforward and requires the use of Lyapunov's methods because the Routh and Nyquist criteria are no longer applicable to these type systems. The first method or indirect method of Lyapunov is so named because it requires the solution to the non-linear differential equation. This method investigates the stability in a small region about each of the equilibrium states of the system. However, knowledge of this type stability is usually not enough in certain problems of interest. The second method or direct method is a tool which enables one to obtain more stability information than is obtainable by use of the first method. The second method of A. M. Lyapunov, published in Russian in 1892, was developed for the purpose of studying the stability of mechanical systems. It was not applied to electrical systems until 1944. The second method is applicable to linear (time invariant and

time varying) and non-linear systems. For linear, time invariant systems, it can be shown that the Routh criterion and the second method impose the same requirements for stability.

As mentioned, the second method is not as straightforward as the Nyquist or Routh criteria. Stability, by this second method, is assured provided there exists a so called Lyapunov function which behaves in a prescribed manner. The determination of the stability of a system consists of a search for a non-unique Lyapunov function,  $V(\underline{x}, t)$ , which is a function of the system states or a function of the states and time. The Lyapunov function is often compared to the energy in a system. However, the energy in a system is not necessarily a Lyapunov function because it does not have to be a decreasing function of time as a Lyapunov function. That is, the average energy of a system may be a decreasing function of time, but the instantaneous energy may not be. Such a system would be unstable and thus the energy would not be a valid Lyapunov function. Since Lyapunov functions are not unique, the determination of the limits of stability is not an easy task. One may obtain a valid function that establishes a set of limits for stability, but nothing can be said about the stability beyond these limits. That is, the function assures stability within the limits but does not assure instability beyond. In the literature there are two basic methods proposed for generating Lyapunov functions, the gradient method and the Lur'e method. As might be suspected, since both methods are straightforward, their requirements on the system are too restrictive. Because of this, and the fact that all non-linear systems cannot be treated in the same way, non-linear systems are broken down into classes depending upon the type of non-linearity.

This enables one to establish a stability criteria that is not so restrictive. In this chapter stability criteria that have been developed for linear systems with a time varying gain in the single feedback loop are considered. The stability of optimal systems is also discussed. Before doing this, it is necessary to state the types of stability which will be of concern.

3.2 Stability Definitions. Before stating the applicable stability definitions, the terminology used in stability analysis of non-linear and time varying systems is reviewed. A non-linear or time varying system is described by the following state equations:

$$\dot{\underline{x}} = f(\underline{x}, t) \quad (3-1)$$

where  $f$  is a non-linear function of the states and time. The solution of Equation (3-1) is written as

$$\Phi(t; \underline{x}_0, t_0) \quad (3-2)$$

where  $\underline{x}_0$  is the state vector at time  $t_0$  or

$$\Phi(t_0; \underline{x}_0, t_0) = \underline{x}_0 \quad (3-3)$$

The equilibrium states,  $\underline{x}_e$ , of Equation (3-1) are all values of  $\underline{x}_e$  such that

$$f(\underline{x}_e, t) = 0 \quad (3-4)$$

Non-linear systems may have one or more equilibrium states and each may be shifted to the origin ( $\underline{x}_e = \underline{0}$ ) by an appropriate change in state coordinates. The norm of a point or vector in the state space is a function which assigns to that point a real number such that

$$(1) \quad \|\underline{x}\| \geq 0 \quad \text{for all } x$$

$$(2) \quad \|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\| \quad \text{for all } \underline{x}, \underline{y}$$

$$(3) \quad \|\alpha \underline{x}\| = |\alpha| \cdot \|\underline{x}\| \quad \text{for all } \underline{x} \text{ and constants } \alpha$$

$$(4) \quad \|\underline{x}\| = 0 \quad \text{implies } \underline{x} = 0$$

A motion is defined as the trajectory starting at any point in the state space. There are many types of stability; a system may be stable in the whole state space or stable only in certain regions. The system may be stable in the sense that the state vector decays to the origin, or in the sense that the state vector is bounded by a finite region. The many variations of stability are discussed by Kalman and Bertram<sup>17</sup> and by Hahn<sup>15</sup>. For purposes of this chapter the following definitions are applicable and are taken from Reference 17.

Definition One<sup>17</sup>. An equilibrium state  $\underline{x}_e$  of a free dynamic system is stable (in the sense of Lyapunov) if for every  $\epsilon > 0$  there exists a real number  $\delta(\epsilon, t_0) > 0$  such that  $\|\underline{x}_0 - \underline{x}_e\| \leq \delta$  implies

$$\|\Phi(t; \underline{x}_0, t_0) - \underline{x}_e\| \leq \epsilon \quad \text{for all } t \geq t_0.$$

If  $\delta$  is not a function of  $t_0$  the system is uniformly stable. Thus, for a system which is stable in the sense of Lyapunov, the state vector is confined to a finite region.

Definition Two<sup>17</sup>. An equilibrium state  $\underline{x}_e$  of a free dynamic system is asymptotically stable if

(1) It is stable and

(2) Every motion starting sufficiently near  $\underline{x}_e$  converges to  $\underline{x}_e$  as  $t \rightarrow \infty$ . Asymptotic stability is of more practical importance

than stability in the sense of Lyapunov. However, an asymptotically stable system may be an impractical design since the system may operate outside of the asymptotically stable region. The ultimate stability desired is asymptotic stability in the large which is asymptotic stability in the whole of state space. The main stability theorem of Lyapunov's second method is stated in Reference 17. This theorem establishes the requirements on the Lyapunov function for uniform asymptotic stability in the large. This is the most stringent of all the types of stability and, as a result, the requirements for the weaker forms of stability are obtained by weakening the conditions of the main theorem. With these few remarks about Lyapunov's second method, the various stability criteria for linear systems with a time varying gain in the feedback loop are investigated.

3.3 Brockett and Fors's Stability Criterion. The system shown in Figure 3-1 has received a great deal of attention because of its appearance in control systems and in electronic devices. It might represent a control system, a parametric amplifier, or other electronic device. Many authors have attempted to develop a stability criterion for this type system which establishes stability limits beyond those of the presently known criteria. Brockett and Fors<sup>11</sup> have established a criterion for stability in the sense of Lyapunov (Definition One) for the system of Figure 3-1. Since the system is linear, although time varying, stability in the sense of Lyapunov implies boundedness for any initial conditions. Their criterion is based on the following theorem which is obtained from the main theorem of Lyapunov's second method.

Theorem One. The time varying system in Figure 3-1 is stable in the sense of Lyapunov if there exists a continuous positive definite function  $V(\underline{X})$ , having continuous first partial derivatives with respect to  $\underline{X}$ , such that the time derivative of  $V(\underline{X})$  is non-positive along any solution of the system.

Before stating the Brockett and Fory criterion, a few remarks concerning the development are in order. The system of Figure 3-1 is expressed in Normal or phase variable form such that the output is written as

$$q(D) X = Y \quad (3-5)$$

where  $D$  is the familiar differential operator. The differential equation describing the system becomes

$$p(D) X + f(t) q(D) X = 0 \quad (3-6)$$

Brockett asserts that if all solutions of Equation (3-6) are bounded, then the system of Figure 3-1 is also bounded. The stability criterion for the system is obtained from a study of Equation (3-6). The development is also based on the analogy between the systems of Figures 3-1 and 3-2. That is, the system in Figure 3-1 is equivalent to a passive network with a time varying resistor across one of the ports. Based on this analogy, it seems reasonable to suspect that Equation (3-6) is stable if  $q(D)/p(D)$  is positive real and  $f(t)$  is non-negative. This analogy led Brockett to the following theorem:

Theorem Two. The system in Figure 3-1 is stable in the sense of Lyapunov provided  $G(s) + \frac{1}{K}$  is a positive real function whose real part is not identically zero and  $0 \leq f(t) \leq K$ .

Theorem Two is a sufficient, but not necessary, condition. This theorem, although useful, severely restricts the linear part of the system. The Nyquist criterion, when applied to the same system with a time invariant gain, predicts stability provided the Nyquist plot does not intersect the negative real axis to the left of  $-\frac{1}{K}$ . However, the above theorem predicts stability if the Nyquist plot avoids the entire plane to the left of  $-\frac{1}{K}$ . Because of these severe restrictions, Brockett attempted to lessen them by placing restrictions on  $\dot{f}(t)$ . The attempt led to the following theorem:

Theorem Three. The system in Figure 3-1 is stable in the sense of Lyapunov if there exists an  $\alpha$ ,  $\beta$ , and  $K$  such that

$$(1) \quad [G(s) + 1/K] \left[ \frac{1 + \alpha s}{1 + \beta s} \right] \text{ is positive real} \quad (3-7)$$

$$(2) \quad 0 \leq f(t) < K \quad (3-8)$$

$$(3) \quad \frac{\dot{f}(t)}{f(t)} \leq 2 \left[ 1 - f(t)/K \right] \min \left( \frac{1}{\alpha}, \frac{1}{\beta} \right) \quad (3-9)$$

This theorem provides sufficient conditions but not necessary conditions. This is to be expected since Lyapunov functions are not unique. To obtain the meaning of the above theorem it is necessary to interpret the equations and the constants  $\alpha$  and  $\beta$ . For the moment, consider  $\alpha$  and  $\beta$  as an arbitrary constant and zero respectively. The positive real requirement of Equation (3-7) is easily established by introduction of the modified polar plot of Popov. This plot, shown in Figure 3-3, is the same as the polar plot except the ordinate is  $w\text{Im}G(jw)$  instead of  $\text{Im}G(jw)$ . Equation (3-7) is positive real provided

$$\operatorname{Re} \left[ \left( G(j\omega) + \frac{1}{K} \right) (1 + \alpha j\omega) \right] \geq 0 \quad (3-10)$$

or

$$\operatorname{Re} G(j\omega) + \frac{1}{K} - \alpha \omega \operatorname{Im} G(j\omega) \geq 0 \quad (3-11)$$

However, the equality portion of Equation (3-11)

$$\operatorname{Re} G(j\omega) + \frac{1}{K} - \alpha \omega \operatorname{Im} G(j\omega) = 0 \quad (3-12)$$

plots as a straight line with a real axis intercept of  $\frac{-1}{K}$  and slope  $\frac{1}{\alpha}$  on the modified polar plot. Therefore, the inequality of Equation (3-11) will hold only if the modified plot of  $G(j\omega)$  lies to the right of this line. Note that the above requirements only satisfy the non-negative real part restriction for a positive real function. Similarly, for a zero  $\alpha$  and a non-zero  $\beta$ , Equation (3-7) is positive real provided the modified polar plot of  $G(j\omega)$  lies to the right of the line

$$\operatorname{Re} G(j\omega) + \frac{1}{K} + \beta \omega \operatorname{Im} G(j\omega) = 0 \quad (3-13)$$

This line is also plotted in Figure 3-4. To apply the theorem, either  $\alpha$  or  $\beta$  is selected to be zero. Both could be zero but the criterion would then reduce to Theorem Two. The selection depends on the modified polar plot of  $G(j\omega)$ . If the modified polar plot of  $G(j\omega)$  crosses the negative real axis with positive slope,  $\beta$  is selected to be zero. If the plot crosses with negative slope,  $\alpha$  is selected to be zero. This theorem allows one to reduce the restrictions on  $f(t)$  by placing restrictions on  $\dot{f}(t)$ . This is the purpose of  $\alpha$  and  $\beta$ . That is, for a system that has a zero  $\beta$ , the maximum restriction on  $f(t)$  occurs for a zero  $\alpha$ . Equation (3-9) then implies that  $\dot{f}(t)$  is not bounded because  $\frac{1}{\alpha}$  is not bounded. However, as  $\alpha$  is increased

Equation (3-7) will be positive real for larger values of  $K$  and thus  $f(t)$ . But  $\dot{f}(t)$  must then take on smaller values as seen from Equation (3-9). There is also an upper limit that  $f(t)$  cannot exceed and this is established by a line tangent to the modified polar plot at the negative real axis intercept. The slope of this tangent line puts an upper limit on  $\mathcal{L}$  and thus establishes the most restrictive set of bounds for  $\dot{f}(t)$ . Equation (3-9) is used to establish the most restrictive set of bounds for  $\dot{f}(t)$  in the following manner. After the maximum value of  $\mathcal{L}$  is determined as explained above, the upper bound on  $\dot{f}(t)$  is determined from Equation (3-9) by substituting the maximum  $\mathcal{L}$  and the maximum  $K$  into the equation. The maximum  $K$  will be the reciprocal of the negative real axis intercept of the modified polar plot of  $G(j\omega)$ . Equation (3-9), for these values of  $\mathcal{L}$  and  $K$  reduces to

$$\dot{f}(t) \leq 2 f(t) \left[ 1 - \frac{f(t)}{K_{\max}} \right] \left[ \frac{1}{\mathcal{L}_{\max}} \right] \quad (3-14)$$

Up to this point the maximum restrictions on  $f(t)$  and  $\dot{f}(t)$  have been established. A compromise between these two extremes is possible for the proper choice of  $\mathcal{L}$  and  $K$ . This is the significant contribution of the theorem. The following example illustrates the explanation above.

Example 3.1 Consider the system in Figure 3-1 where

$$G(s) = \frac{1}{s^2 + ps} \quad (3-15)$$

The differential equation describing the system is

$$\ddot{X} + p\dot{X} + f(t) X = 0 \quad (3-16)$$

The modified polar plot of  $G(j\omega)$ , shown in Figure 3-5, is a straight line passing through the origin with slope  $p$ . The problem is to determine the range of values of  $f(t)$  and  $\dot{f}(t)$  for which the system is stable. Since the plot crosses the real axis with positive slope,  $\beta$  is taken to be zero and the equations of Theorem Three reduce to

$$[G(s) + 1/K] (1 + \mathcal{L}s) \quad (3-17)$$

$$0 \leq f(t) < K \quad (3-18)$$

$$\frac{\dot{f}(t)}{f(t)} \leq -\frac{2}{\mathcal{L}} [1 - f(t)/K] \quad (3-19)$$

To obtain the maximum restriction of  $f(t)$  an  $\mathcal{L}$  equal to zero is always chosen. For this choice of  $\mathcal{L}$ , Equation (3-16) is positive real if

$$K \leq p^2 \quad (3-20)$$

Since the maximum value that  $K$  can assume in order that

$$G(s) + 1/K \quad (3-21)$$

is positive real is the minimum of the real part of  $G(j\omega)$ , Equation (3-20) can be obtained from the modified polar plot. To obtain the maximum value of  $K$ , a vertical line tangent to the modified polar plot is constructed. The intersection of this line with the negative real axis yields  $\frac{1}{K_{\max}}$ . Note that this line has slope

$$\frac{1}{\mathcal{L}} = \infty \quad (3-22)$$

and the modified polar plot of  $G(j\omega)$  lies to the right of it. Thus, this line represents the choice of  $\mathcal{L}$  and  $K$  such that  $f(t)$  has the maximum restrictions and  $\dot{f}(t)$  has none. Equations (3-18) and (3-19) reduce to

$$0 \leq f(t) < p^2 \quad (3-23)$$

$$\dot{f}(t) \leq \infty \quad (3-24)$$

To place the maximum restrictions on  $\dot{f}(t)$  and the least restrictions on  $f(t)$ , the associated  $\mathcal{L}$  and  $K$  are obtained from the line tangent to the modified polar plot of  $G(j\omega)$  at the real axis intercept of the plot. As required by Theorem Three, the plot must lie on or to the right of this line. The slope of this line determines the maximum value of  $\mathcal{L}$ , and thus the restrictions on  $\dot{f}(t)$ , and the maximum value of  $K$ . Equations (3-18) and (3-19), for this choice of  $\mathcal{L}$  and  $K$ , reduce to

$$0 \leq f(t) < \infty \quad (3-25)$$

$$f(t) \leq 2pf(t) \quad (3-26)$$

The lines representing these two extremes are shown in Figure 3-5. In between these two lines is a compromise line (dotted). This line represents the tradeoff between the two sets of restrictions. The value of  $\mathcal{L}$  and  $K$  to determine the restrictions on  $f(t)$  and  $\dot{f}(t)$  are obtained from the slope and real axis intercept of the compromise line which can vary between the two extremes. The following steps illustrate the application of Theorem Three.

- (a) Plot the modified polar plot of  $G(j\omega)$ .
- (b) From the plot determine whether  $\mathcal{L}$  or  $\beta$  should be zero.
- (c) Construct the line representing the maximum restrictions on  $f(t)$  by constructing a vertical line tangent to the modified polar plot of  $G(j\omega)$  such that the plot lies to the right of the vertical line.
- (d) Construct the line representing the maximum restrictions on  $\dot{f}(t)$  by constructing a line tangent to the modified polar plot of  $G(j\omega)$  at the real axis intercept such that the plot lies to the right of the line.

- (e) All the possible compromises lie in between the two extreme lines. Any compromise line is valid provided the modified polar plot of  $G(j\omega)$  lies to the right of it.
- (f) The values of  $\alpha$  and  $K$  are obtained from the slopes and real axis intercepts of these lines.

In the event that the time varying gain in the feedback loop has an  $\dot{f}(t)$  that is not bounded, one must utilize Theorem Two to obtain the range of values of  $f(t)$  for which the system is stable. Brockett has developed a stability criterion which is identical to Theorem Two but is easier to apply because it uses the ordinary Nyquist plot.

3.4 Brockett's Circle Criterion<sup>12</sup>. The circle criterion establishes the range of values of  $f(t)$  which will assure stability assuming no restrictions on  $f(t)$ . The significance of this criterion is twofold. It is easy to apply and it reduces to the Nyquist criterion as a special case. The circle criterion is contained within the following theorem:

Theorem Four<sup>12</sup> Let  $q(s)$  and  $p(s)$  be polynomials without common factors and let

$$\dot{\underline{x}} = A \underline{x} + B u \quad (3-27)$$

$$u = -f(t) C^T \underline{x} \quad (3-28)$$

$$y = C^T \underline{x} \quad (3-29)$$

be an irreducible representation of  $G(s) = q(s)/p(s)$  (system is controllable and observable). Then, if  $p(s)$  has no zeros in the right half plane

- (a) All solutions of Equation (3-27) are bounded if

$$0 \leq \beta \leq f(t) \leq \alpha \quad (3-30)$$

and the Nyquist locus of  $G(s)$  does not encircle or intersect the open disk which is centered on the negative real axis of the  $G(s)$  plane and has as a diameter the segment of the negative real axis  $(-\frac{1}{\mathcal{L}}, -\frac{1}{\beta})$

- (b) All solutions are bounded and go to zero at an exponential rate if there is some  $\epsilon > 0$  such that

$$0 \leq \beta + \epsilon \leq f(t) \leq \mathcal{L} - \epsilon \quad (3-31)$$

and the Nyquist locus behaves as in (a).

This theorem is also concerned with the time varying system of Figure 3-1. The open disk is shown in Figure 3-6. Part (a) of the criterion simply says that if the Nyquist plot of  $G(s)$  does not intersect or encircle the disk of radius  $1/2 (\frac{1}{\mathcal{L}} - \frac{1}{\beta})$  and center at  $-3/2\mathcal{L} + 1/2\beta$  then the system is stable in the sense of Lyapunov. Note that if the lower limit on  $f(t)$ ,  $\beta$ , is zero then the disk becomes the entire plane to the left of  $-1/\mathcal{L}$ . This is the same result as was obtained using Theorem Two. To show that this criterion reduces to the Nyquist criterion, consider  $f(t)$  to be a constant  $K$ . Then the open disk reduces to a point at  $-1/K$ . The system is stable, provided the Nyquist plot of  $G(s)$  does not intersect or encircle the point  $-1/K$ . But this is simply the Nyquist criterion. Part (b) of the criterion implies that if one constructs the largest disk possible that is not intersected or encircled by the Nyquist plot  $G(j\omega)$  and can find an  $\epsilon > 0$  that will translate the  $\mathcal{L}$  and  $\beta$  associated with the largest disk to the actual  $\mathcal{L}$  and  $\beta$  of  $f(t)$ , then all solutions go to zero at an exponential rate. This is asymptotic stability and not just bounded output stability. To demonstrate the use of the circle criterion consider the following example:

Example 3.2 The open loop plant of Figure 3-1 has the transfer function

$$G(s) = \frac{1}{S(S+2)} \quad (3-32)$$

and the Nyquist plot is shown in Figure 3-7. The time varying gain has the following limits:

$$0.1 \leq f(t) \leq 2 \quad (3-33)$$

and the circle criterion disk for this  $f(t)$  is also plotted in Figure 3-7. Since  $G(j\omega)$  does not intersect or encircle this disk, the system is stable. In fact, the system is stable for the following range of values for  $f(t)$ :

$$0 \leq f(t) \leq 4 \quad (3-34)$$

These limits are established by constructing the largest disk subject to the requirements of the theorem. This circle is the entire plane to the left of line

$$\min \operatorname{Re} G(j\omega) = -0.25 \quad (3-35)$$

This line corresponds to an  $\alpha$  of value 4. It is obvious that the system is not asymptotically stable because there is no  $\epsilon > 0$  that will translate the  $\alpha$  and  $\beta$  of the infinite circle to the actual  $\alpha$  and  $\beta$  of  $f(t)$ .

The previous sections of this chapter have considered the problem of stability for time varying systems. The various stability criteria for these systems were developed primarily from the stability theorems of Lyapunov's second method. At first glance it might appear that the design of controls systems by the optimization technique eliminates the stability problem. This is not true. Kalman, in Reference 3, makes this clear by the statement, "Optimality does not imply stability!"

One might also question the relation between non-linear (and time varying) systems and the design of linear systems by the optimal control theory. The answer to this question is that the optimal control approach, more often than not, leads to a non-linear or time varying system. The next section considers the problem of stability as it relates to optimal control.

3.5 Stability and Optimal Controls. Lyapunov stability theory is very similar to optimal control theory in many respects. The formulation of the optimization problem consists essentially of selecting the cost function that expresses the desired objective. This function may be dependent upon the system states, time, the control, or all three. The choice of the index is not an easy one to make. However, assuming that the choice has been made, this function can usually be translated into the state space as a family of constant cost surfaces. The solution to the optimization problem is the control that produces system state trajectories that transit these surfaces in a manner such that they are always headed toward a lesser cost surface. The determination of stability consists of a search for a Lyapunov function which may be dependent upon the system states and time. If this function and its time derivative behave in a prescribed manner, then stability is assured. The Lyapunov function also may be translated into the state space as a family of constant value surfaces. If the unaltered system state trajectories transit these surfaces in a manner such that the Lyapunov is a decreasing function of time, then stability is assured. But this is where the similarity ends. As emphasized by Kalman, the optimal control approach may lead to an unstable system unless some sort of stability consideration is incorporated into the optimization

problem. The obvious place to inject this consideration is in the choice of performance index. That is, to select a performance index that is a valid Lyapunov function for the optimal system. For instance, if the integral squared error criterion is chosen for the free ( $\mu = 0$ ) linear regulator then this criterion is a Lyapunov function for the optimal system provided the squared error criterion does not vanish along any state trajectory of the optimal system. This result is crystalized in the following theorem by Kalman.

Theorem Five<sup>17</sup> Consider a free ( $\mu = 0$ ), linear, time invariant dynamic system with an equilibrium state at the origin and assume

(a) The error criterion  $L(\underline{x})$  is positive definite and  $L(\underline{0}) = 0$ .

(b) The performance index

$$J(\underline{X}) = \int_0^{\infty} L(\underline{x}) dt \quad (3-36)$$

is finite in some neighborhood of the origin.

Then the equilibrium state is asymptotically stable.

This theorem is applicable to the parameter optimization problem, which selects the value of a system parameter that minimizes the performance index. Stability of this type of system is assured by placing certain restrictions on the cost function that requires it to be a valid Lyapunov function for the optimal system. It is interesting to note that the error criterion does not have to be quadratic. The positive definite requirement on  $L(\underline{x})$  ensures that the cost function or the Lyapunov function does not vanish anywhere other than at the origin. If it did vanish, the optimal system would have an equilibrium state other than the origin. This is the reason why the Q and P

matrices of the performance index of Chapter II were required to be positive semi-definite and positive definite, respectively.

The above theorem is applicable to the unforced linear regulator but not to the true optimization of the forced linear regulator for which the optimal control is derived. This problem may have a finite control time and thus a new stability criteria is required. Kalman has considered this problem and with the following results.

Theorem Six<sup>3</sup>. If the assumptions

- (a) The pair  $(A, L^T)$ , where  $Q = LL^T$  and  $A$  is the  $A$  matrix of the plant state equations, is observable.
- (b) The plant is controllable

are satisfied then  $R(t = \infty)$  is positive definite and the optimal control law is stable.

The first assumption of this theorem ensures that the integrand of the cost function

$$\underline{x}^T Q \underline{x} + \underline{u}^T P \underline{u} \quad (3-37)$$

does not vanish along the optimal trajectories. This implies that the cost function is a Lyapunov function for the optimal system. Since the evaluated cost function is

$$J = \underline{x}^T R \underline{x} \quad (3-38)$$

then  $R$  must be positive definite. The second assumption guarantees a solution,  $R$ , to the matrix Ricatti differential equation.

This chapter has considered the problem of stability for time varying systems and the problem of stability for optimal systems. The similarity of stability theory and optimal theory suggests that optimal

control theory might be useful in establishing stability criteria for non-linear and time varying systems. The linear regulator problem with finite control time suggests such an idea. For that particular problem the feedback gains are time varying but stability is still assured if the requirements of Theorem Six are met. The optimal controls approach would be useful not for the purpose of designing an optimal system but for its ability to yield non-linear and time varying systems. The optimal approach might provide a refreshing look at the stability of these systems.

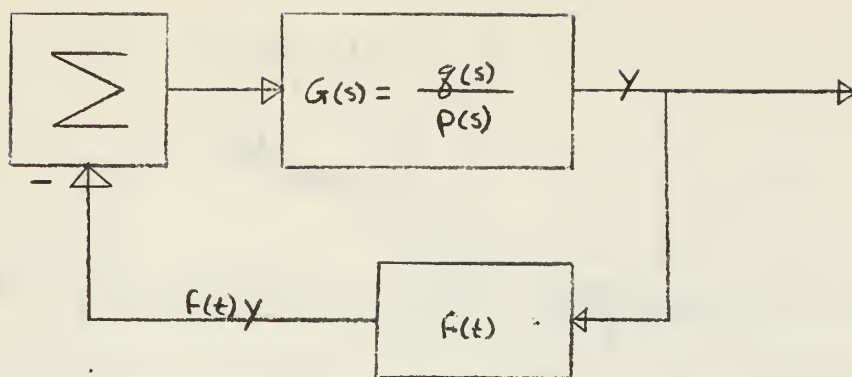


Figure 3-1. Block Diagram of Time Varying System

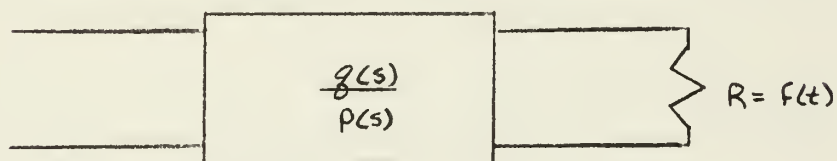


Figure 3-2. Network Equivalent of Figure 3-1

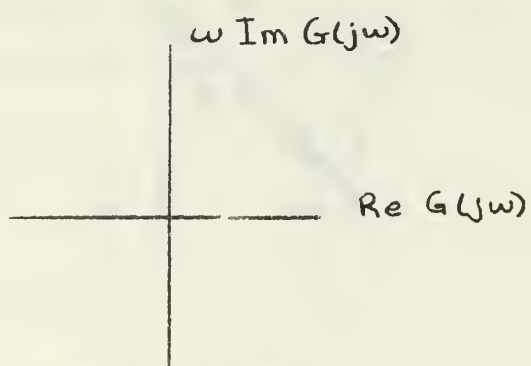


Figure 3-3. Axes of Modified Polar Plot

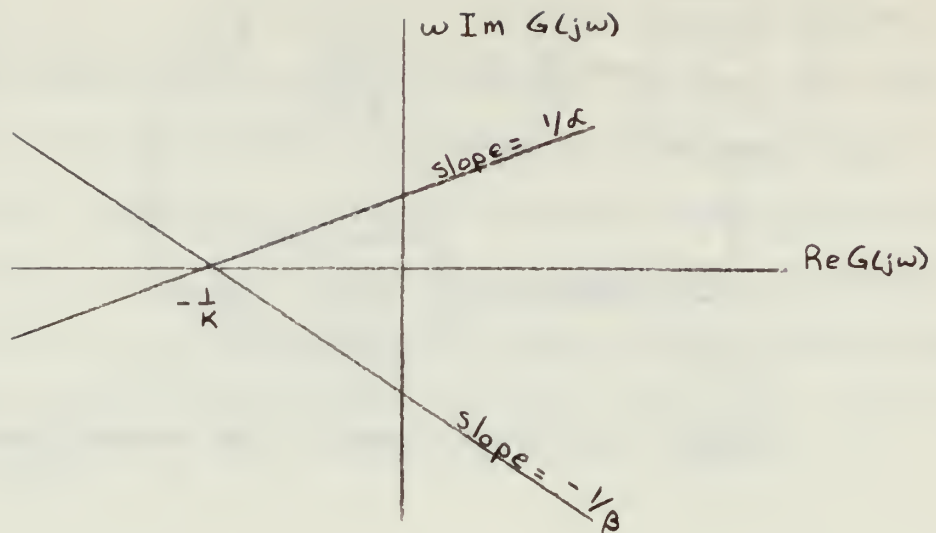


Figure 3-4. Modified Polar Plot

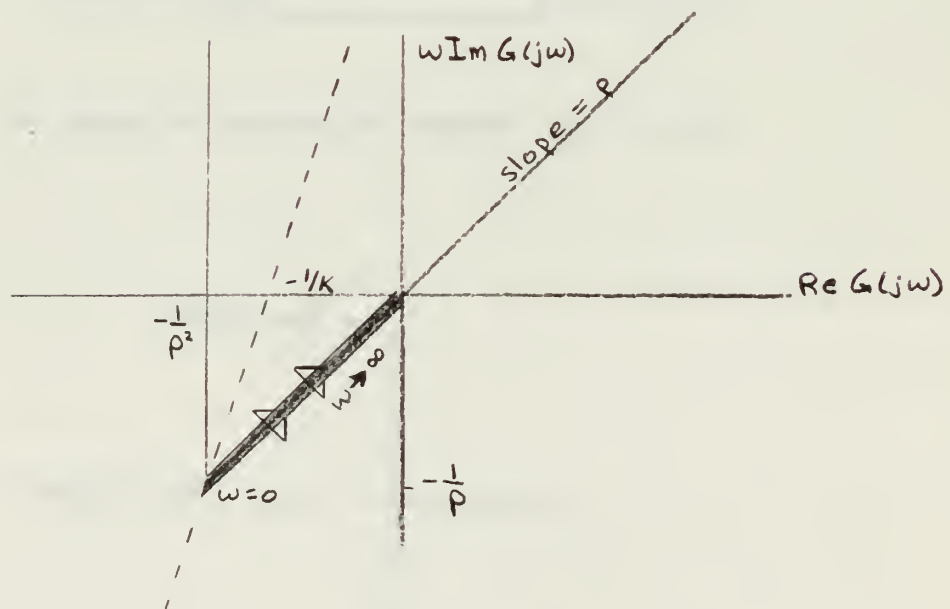


Figure 3-5. Modified Polar Plot of  $G(s) = \frac{1}{s^2 + ps}$

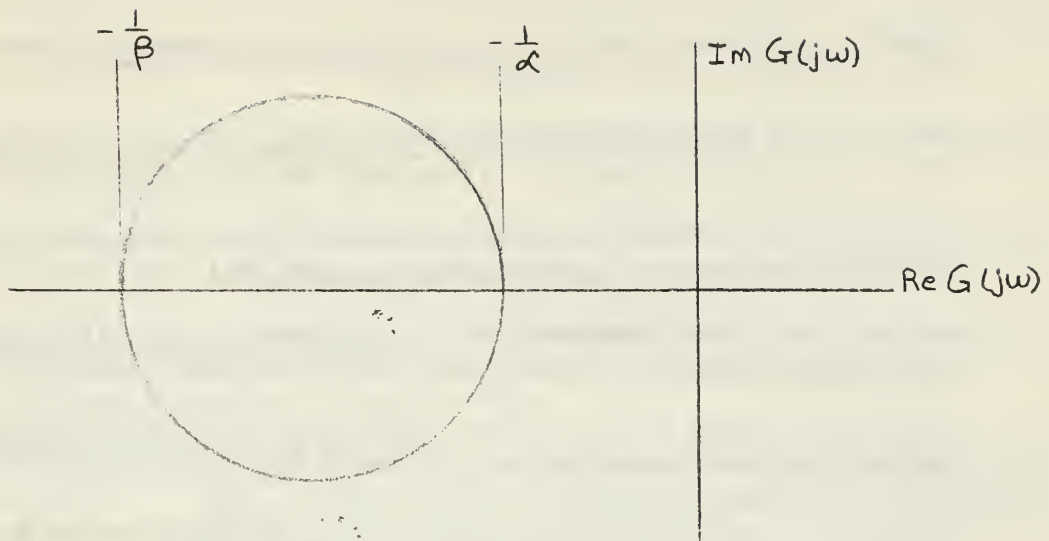


Figure 3-6. Circle Criterion Disk

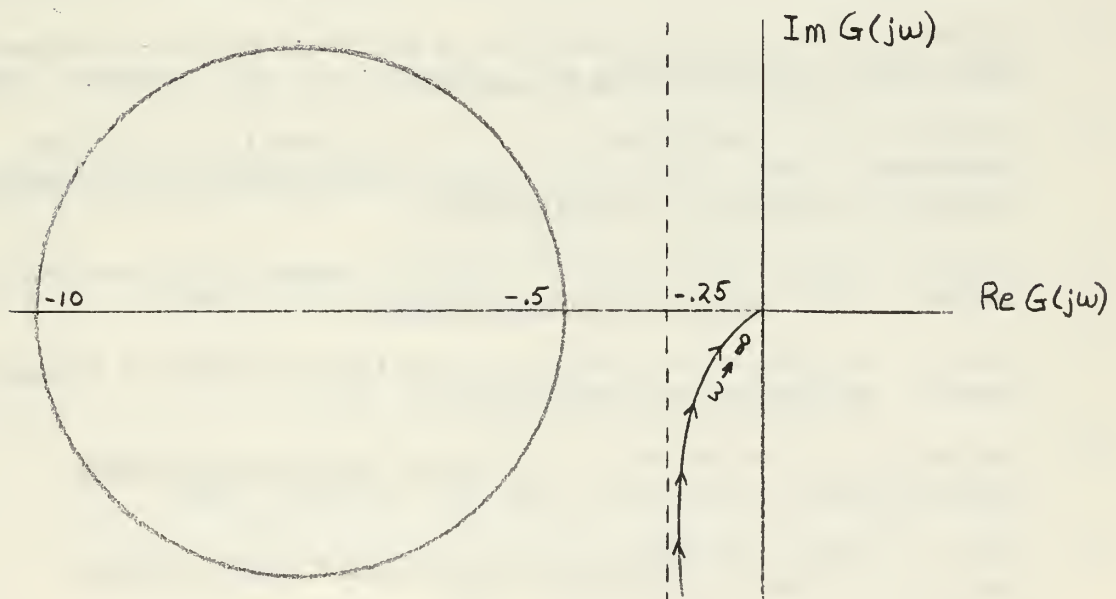


Figure 3-7. Circle Criterion for Example 3.2

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## APPENDIX A

A.1 The following is a list of the state equations of the networks and machines of Figure A-1 through A-7 respectively.

(1) Phase Lag Network

$$\dot{V}_c = \left[ \frac{-1}{RC} \right] V_c + \left[ \frac{1}{RC} \right] E_{in} \quad (A-1)$$

$$E_o = [1] V_c \quad (A-2)$$

(2) Phase Lag Network with High Frequency Attenuation

$$\dot{V}_c = \left[ \frac{-1}{C(R_1 + R_2)} \right] V_c + \left[ \frac{1}{C(R_1 + R_2)} \right] E_{in} \quad (A-3)$$

$$E_o = \left[ \frac{R_1}{R_1 + R_2} \right] V_c + \left[ \frac{R_2}{R_1 + R_2} \right] E_{in} \quad (A-4)$$

(3) Phase Lead Network

$$\dot{V}_c = \left[ -\frac{1}{RC} \right] V_c + \left[ \frac{1}{RC} \right] E_{in} \quad (A-5)$$

$$E_o = [-1] V_c + [1] E_{in} \quad (A-6)$$

(4) Phase Lead Network with Fixed DC Attenuation

$$\dot{V}_c = \left[ -\frac{R_1 + R_2}{R_1 R_2 C} \right] V_c + \left[ \frac{1}{R_2 C} \right] E_{in} \quad (A-7)$$

$$E_a = [1]V_a + [1]E_{in} \quad (A-8)$$

- (5) DC Motor with Constant Field Current and Negligible Armature Inductance

$$\begin{bmatrix} \dot{\Theta}_m \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{K_T K_m}{J R_a} \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_T}{J R_a} \end{bmatrix} E_{in} \quad (A-9)$$

$$\Theta_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} \quad (A-10)$$

- (6) DC Motor with Constant Armature Current

$$\begin{bmatrix} \dot{I}_f \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 \\ \frac{K_T}{J} & -\frac{f}{J} \end{bmatrix} \begin{bmatrix} I_f \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \end{bmatrix} E_{in} \quad (A-11)$$

$$\dot{\Theta}_m = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_f \\ \dot{\Theta}_m \end{bmatrix} \quad (A-12)$$

- (7) Two Phase Servomotor

$$\begin{bmatrix} \dot{\Theta}_m \\ \ddot{\Theta}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{(K_n + f)}{J} \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_e}{J} \end{bmatrix} E_{in} \quad (A-13)$$

$$\Theta_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Theta_m \\ \dot{\Theta}_m \end{bmatrix} \quad (A-14)$$

The following is a list defining the constants and variables of the above state equations.

$K_m$	=	motor back emf constant
$K_t$	=	motor torque constant
$R_a$	=	armature resistance
$R_f$	=	field resistance
$L_f$	=	armature inductance
$I_a$	=	armature current
$I_f$	=	field current
$E_{in}$	=	applied armature or field voltage
$\theta_m$	=	motor shaft angular position
$T$	=	gear box ratio =
$J_m$	=	motor inertia
$J_L$	=	load inertia
$J$	=	$J_m + T^2 J_L$
$f_m$	=	motor friction
$f_L$	=	load friction
$f$	=	$f_m + T^2 f_L$
$K_e$	=	motor torque constant
$K_n$	=	motor viscous friction constant

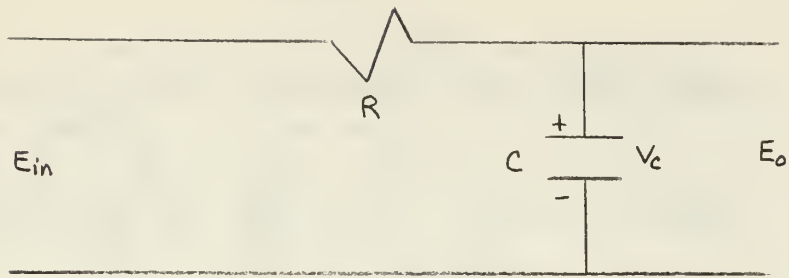


Figure A-1. Phase Lag Network

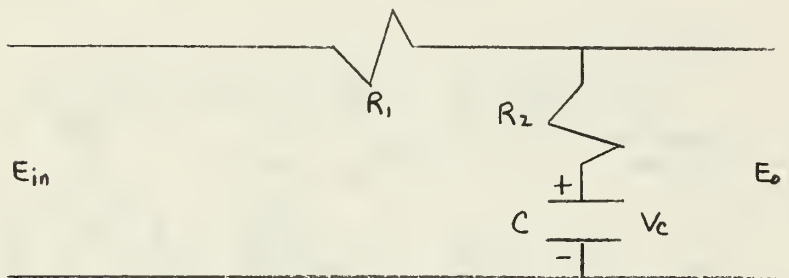


Figure A-2. Phase Lag Network with High Frequency Attenuation

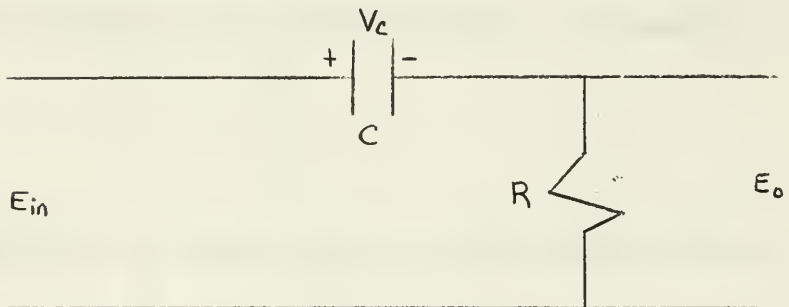


Figure A-3. Phase Lead Network

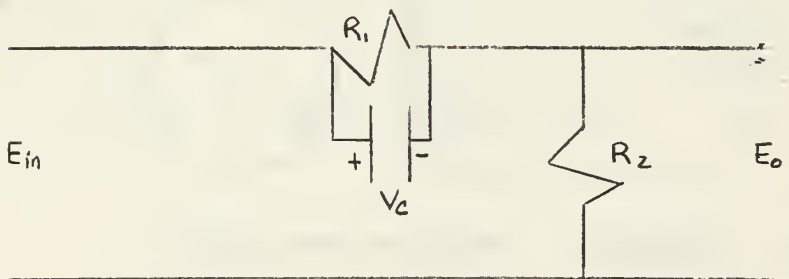


Figure A-4. Phase Lead Network with Fixed DC Attenuation

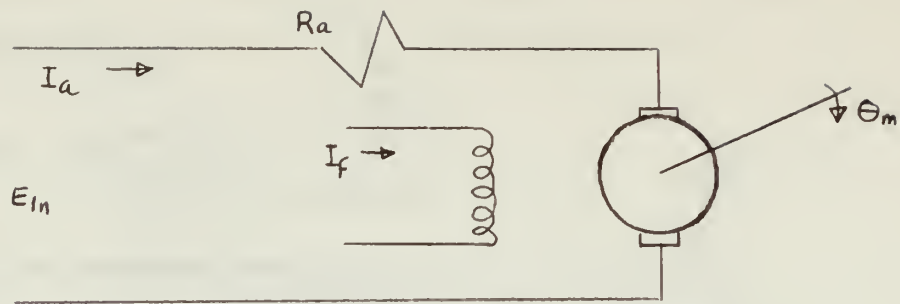


Figure A-5. DC Motor with Constant Field Current and Negligible Armature Inductance

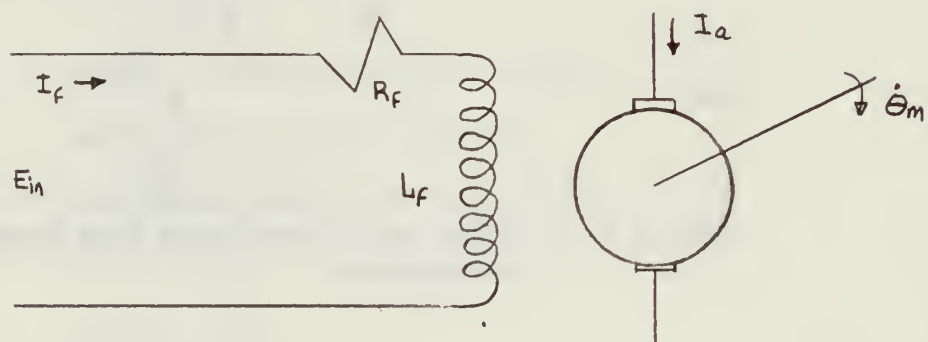


Figure A-6. DC Motor with Constant Armature Current

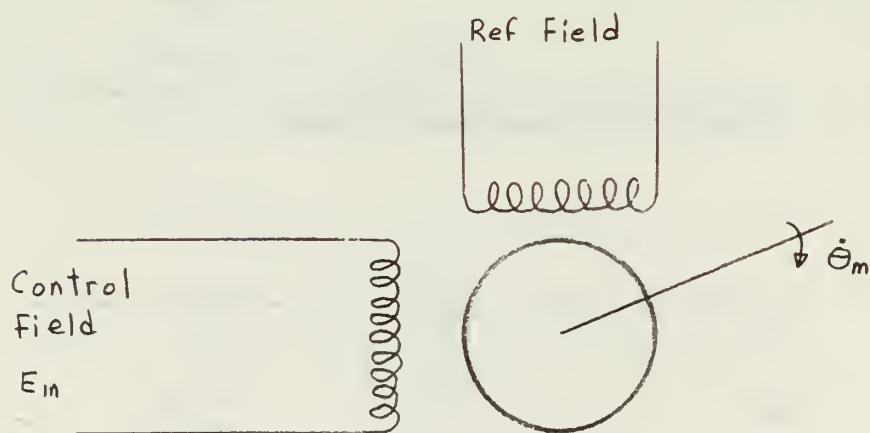


Figure A-7. Two Phase Servo Motor

## APPENDIX B

### B.1. Derivation of the L Matrix for a Second Order System

$$\int_0^{\infty} \underline{X}^T Q \underline{X} dt = \int_0^{\infty} [g_{11} \dot{X}_1^2 + g_{22} \dot{X}_2^2] dt = \int_0^{\infty} [g_{11} \dot{X}_1^2 + g_{22} \dot{X}_1^2] dt \quad (B-1)$$

$$= \int_0^{\infty} [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1]^2 dt - \int_0^{\infty} 2\sqrt{g_{11}} \sqrt{g_{22}} \dot{X}_1 \dot{X}_1 dt \quad (B-2)$$

$$= \int_0^{\infty} [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1]^2 dt - \left. \sqrt{g_{11}} \sqrt{g_{22}} X_1^2(t) \right|_{t=0}^{t=\infty} \quad (B-3)$$

$$= \int_0^{\infty} [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1]^2 dt + \sqrt{g_{11}} \sqrt{g_{22}} X_1^2(0) \quad (B-4)$$

From Equation (2-52) the L matrix is

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \sqrt{g_{11}} \\ \sqrt{g_{22}} \end{bmatrix} \quad (B-5)$$

### B.2. Derivation of the L Matrix for a Third Order System

$$\int_0^{\infty} \underline{X}^T Q \underline{X} dt = \int_0^{\infty} [g_{11} \dot{X}_1^2 + g_{22} \dot{X}_1^2 + g_{33} \ddot{X}_1^2] dt \quad (B-6)$$

$$\begin{aligned} &= \int_0^{\infty} [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1 + \sqrt{g_{33}} \ddot{X}_1]^2 dt \\ &\quad - 2 \int_0^{\infty} [\sqrt{g_{11}} \sqrt{g_{22}} \dot{X}_1 \dot{X}_1 + \sqrt{g_{11}} \sqrt{g_{33}} \dot{X}_1 \ddot{X}_1 - \sqrt{g_{22}} \sqrt{g_{33}} \dot{X}_1 \ddot{X}_1] dt \end{aligned} \quad (B-7)$$

$$= \int_0^{\infty} [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1 + \sqrt{g_{33}} \ddot{X}_1]^2 dt + \sqrt{g_{11}g_{22}} \dot{X}_1^2(0) \quad (B-8)$$

$$+ \sqrt{g_{22}g_{33}} \dot{X}_1^2(0) + \int_0^{\infty} 2\sqrt{g_{11}g_{33}} \dot{X}_1 \ddot{X}_1 dt$$

$$= \int_0^{\infty} \left\{ [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22}} \dot{X}_1 + \sqrt{g_{33}} \ddot{X}_1]^2 + 2\sqrt{g_{11}g_{33}} \dot{X}_1^2 \right\} dt \quad (B-9)$$

$$+ \sqrt{g_{11}g_{22}} \dot{X}_1^2(0) + \sqrt{g_{22}g_{33}} \dot{X}_1^2(0)$$

$$= \int_0^{\infty} \left\{ [\sqrt{g_{11}} \dot{X}_1 + \sqrt{g_{22} + 2\sqrt{g_{11}g_{33}}} \dot{X}_1 + \sqrt{g_{33}} \ddot{X}_1]^2 \right\} dt \quad (B-10)$$

$$+ [\sqrt{g_{11}g_{22}} + \sqrt{g_{22} + 2\sqrt{g_{11}g_{33}}}] \dot{X}_1^2(0) + [\sqrt{g_{11}g_{22}} + \sqrt{g_{22} + 2\sqrt{g_{11}g_{33}}}] \dot{X}_1^2(0)$$

From Equation (2-52) the L matrix is

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} \sqrt{g_{11}} \\ \sqrt{g_{22} + 2\sqrt{g_{11}g_{33}}} \\ \sqrt{g_{33}} \end{bmatrix} \quad (B-11)$$

## APPENDIX C

C.1 Mathematical Formulation of the Ricatti Differential Equation for a Third Order Plant. The state equations for the plant of Figure 2-5 are

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -ab & -(a+b) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G \end{bmatrix} u \quad (C-1)$$

Define

$$F = a + b \quad (C-2)$$

$$G = ab \quad (C-3)$$

The matrices required by the Ricatti equation are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -G & -F \end{bmatrix} \quad (C-4)$$

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{12} & R_{22} & R_{23} \\ R_{13} & R_{23} & R_{33} \end{bmatrix} \quad (C-5)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ G \end{bmatrix} \quad (C-6)$$

$$P = P_{11} \quad (C-7)$$

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix} \quad (C-8)$$

The terms of the Ricatti differential equation

$$A^T R + RA - RBP^{-1}B^T R + Q = -\dot{R} \quad (C-9)$$

are

$$A^T R = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_{11} - G\lambda_{13} & \lambda_{12} - G\lambda_{23} & \lambda_{13} - G\lambda_{33} \\ \lambda_{12} - F\lambda_{13} & \lambda_{22} - F\lambda_{23} & \lambda_{23} - F\lambda_{33} \end{bmatrix} \quad (C-10)$$

$$RA = \begin{bmatrix} 0 & \lambda_{11} - G\lambda_{13} & \lambda_{12} - F\lambda_{13} \\ 0 & \lambda_{12} - G\lambda_{23} & \lambda_{22} - F\lambda_{23} \\ 0 & \lambda_{13} - G\lambda_{33} & \lambda_{23} - F\lambda_{33} \end{bmatrix} \quad (C-11)$$

$$-RBP^{-1}B^T R = \begin{bmatrix} -\frac{G^2}{p''} \lambda_{13}^2 & -\frac{G^2}{p''} \lambda_{13} \lambda_{23} & -\frac{G^2}{p''} \lambda_{13} \lambda_{33} \\ -\frac{G^2}{p''} \lambda_{13} \lambda_{23} & -\frac{G^2}{p''} \lambda_{23}^2 & -\frac{G^2}{p''} \lambda_{23} \lambda_{33} \\ -\frac{G^2}{p''} \lambda_{13} \lambda_{33} & -\frac{G^2}{p''} \lambda_{23} \lambda_{33} & -\frac{G^2}{p''} \lambda_{33}^2 \end{bmatrix} \quad (C-12)$$

Substituting the above terms into the Ricatti equation yields the following six first order non-linear differential equations,

$$-\frac{G^2}{p''} \lambda_{13}^2(t) + q_{11} = -\dot{\lambda}_{11}(t) \quad (C-13)$$

$$-\frac{G^2}{p''} \lambda_{13}(t) \lambda_{23}(t) + \lambda_{11}(t) - G\lambda_{13}(t) = -\dot{\lambda}_{12}(t) \quad (C-14)$$

$$-\frac{G^2}{P''} \lambda_{13}(t) \lambda_{33}(t) + \lambda_{12}(t) - F \lambda_{13}(t) = -\dot{\lambda}_{13}(t) \quad (C-15)$$

$$-\frac{G^2}{P''} \lambda_{23}^2(t) + g_{22} + 2(\lambda_{12}(t) - G \lambda_{23}(t)) = -\dot{\lambda}_{22}(t) \quad (C-16)$$

$$-\frac{G^2}{P''} \lambda_{23}(t) \lambda_{33}(t) + \lambda_{13}(t) - G \lambda_{33}(t) + \lambda_{22}(t) - F \lambda_{23}(t) = -\dot{\lambda}_{23}(t) \quad (C-17)$$

$$-\frac{G^2}{P''} \lambda_{33}^2(t) + g_{33} + 2(\lambda_{23}(t) - F \lambda_{33}(t)) = -\dot{\lambda}_{33}(t) \quad (C-18)$$

Note that the number of first order equations which must be integrated to obtain the solution to the Ricatti equation is

$$n \frac{(n+1)}{2} \quad (C-19)$$

where n is the order of the plant. These equations are easily solved on a digital computer by integrating backward in time with the following initial conditions:

$$R(t = 0) = 0 \quad (C-20)$$

The solution to these equations is easily checked by using Equation (C-13). The steady state value of  $\lambda_{13}$  is

$$\lambda_{13} = \frac{1}{G} \sqrt{g'' P''} \quad (C-21)$$

For higher order plants, Equation (C-21) becomes

$$\lambda_{1n} = \frac{1}{G} \sqrt{g'' P''} \quad (C-22)$$

This result is valid if the description of the plant is of Normal Form.

## APPENDIX D

D.1. Block Diagram Manipulations. When control systems are represented by block diagrams, manipulations are often performed to convert the system to a convenient form for analysis. Manipulations such as these often result in a new set of state variables for the system. Thus, a block diagram manipulation is comparable with the changing from one state variable representation to another. For the linear regulator problem, the Normal Form has an advantage over the others because of the simple modeling procedure due to Schultz and Melsa. Whatever representation is selected, one must be aware of the effect of these changes on optimization process. The problem of interest is again the second order linear regulator. In Chapter II this problem was solved by substituting the A matrix of the plant of Figure 2-1. Since there is a certain amount of matrix algebra that must be performed to obtain the solution to the Ricatti equation, the problem is simplified by moving the pole of the plant into the feedback loop. This is a valid operation since it is known that the optimal control is state variable feedback. Figure D-1 shows this manipulation. The optimal feedback gains without this manipulation are represented by  $K_1$  and  $K_2$ . Now, the open loop plant of Figure D-1 is optimized by the same procedure used in Chapter II. The optimal  $k_1$  and  $k_2$  for the new plant are obtained and then  $K_1$  and  $K_2$  are found from the relations

$$K_1 = k_1 \quad (D-1)$$

$$K_2 = k_2 - a/G \quad (D-2)$$

$K_1$  and  $K_2$  are not the same as the  $K_1$  and  $K_2$  obtained in Chapter II.

This is reasonable since the optimization process does not take into account the ability of the actual plant to drive the state vector to the origin. However, the same optimal system will result if the Q and P matrices are also changed for the new plant as follows:

$$\frac{\bar{q}_{11}}{\bar{p}_{11}} = \frac{\hat{q}_{11}}{\hat{p}_{11}} \quad (D-3)$$

$$\frac{\bar{q}_{22}}{\bar{p}_{11}} = \frac{\hat{q}_{22}}{\hat{p}_{11}} + \left(\frac{a}{G}\right)^2 \quad (D-4)$$

The bar indicates the new plant, and the hat indicates the original plant. Note that the  $\bar{q}_{22}/\bar{p}_{11}$ , which represents emphasis on minimizing the state as compared to emphasis on minimizing the control for state  $X_2$ , increases as the square of the quantity added to the feedback loop of state  $X_2$ . The term  $\bar{q}_{11}/\bar{p}_{11}$  did not change because the feedback loop of state  $X_1$  did not change. One might look at this result in the light that, since the pole is now used as control, the emphasis on minimizing the state must be increased over the emphasis on minimizing the control to obtain the same optimal system.

The very same procedure is carried out for the case where both the pole and the open loop gain are moved into the feedback loop. Figure D-2 shows this manipulation. For this case, the Q and P matrices must be changed as follows:

$$\frac{\bar{q}_{11}}{\bar{p}_{11}} = G^2 \frac{\hat{q}_{11}}{\hat{p}_{11}} \quad (D-5)$$

$$\frac{\bar{q}_{22}}{\bar{p}_{11}} = G^2 \left[ \frac{\hat{q}_{22}}{\hat{p}_{11}} + \left(\frac{a}{G}\right)^2 \right] \quad (D-6)$$

These equations are  $G^2$  times the results obtained for the case where only the pole is moved. Thus, the same optimal system is obtained by moving all poles and open loop gains into the feedback loops and then increasing the appropriate  $\hat{q}_{ii}/\hat{p}_{ii}$  by the square of the pole values and then multiplying this result by the square of the open loop gain. This result is useful because it introduces the maximum number of zero elements into the A matrix. It also demonstrates the effect of block diagram manipulations on the optimization problem.

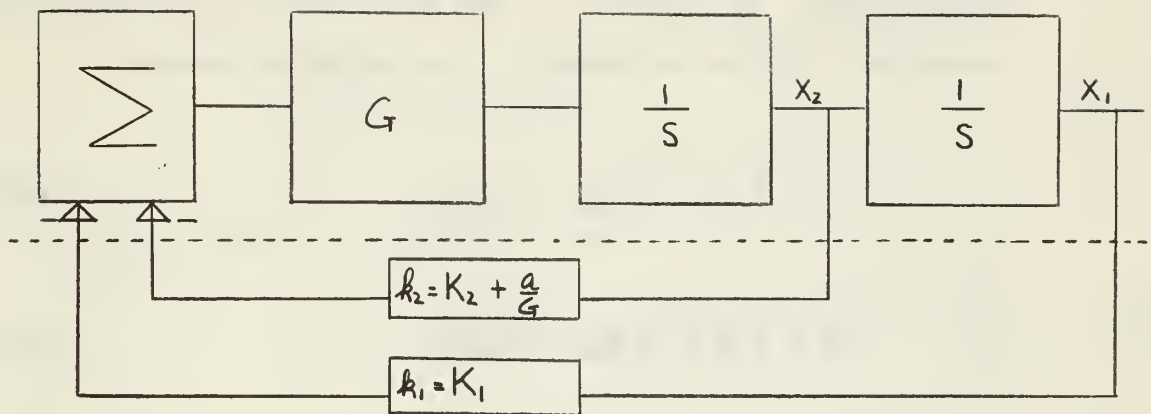


Figure D-1. Optimal System with Pole Moved into Feedback Loop

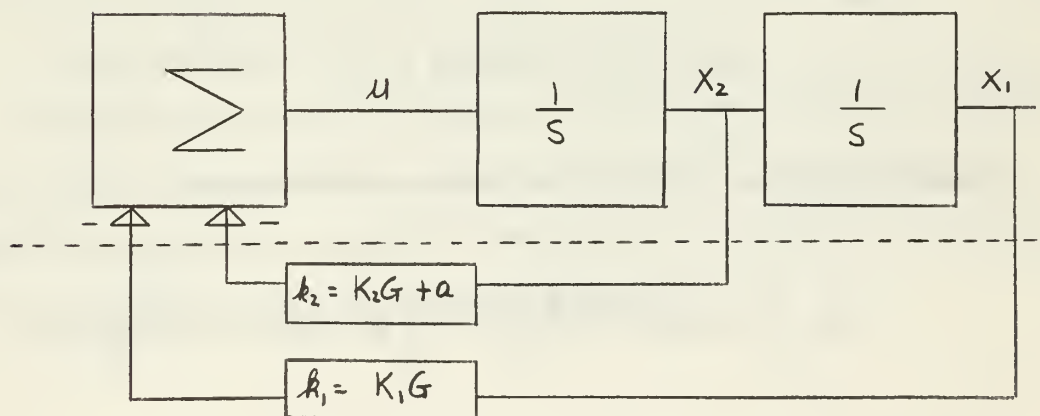


Figure D-2. Optimal System with Pole and Gain Moved into Feedback Loop

## APPENDIX E

E.1 Derivation of the System State Equations for Complex Interconnections. In Chapter I it was stated that the combined state equations could be rearranged in the following manner:

$$\dot{\underline{X}} = \underline{A} \underline{X} + \underline{B}_a \underline{Q}_a + \underline{B}_b \underline{Q}_b \quad (\text{E-1})$$

$$\underline{Y} = \underline{C} \underline{X} + \underline{D}_a \underline{Q}_a + \underline{D}_b \underline{Q}_b \quad (\text{E-2})$$

$$\underline{Q}_b = \underline{\Delta} \underline{Y} \quad (\text{E-3})$$

$\underline{R}_b$  is the vector of component inputs that are not system inputs.

Solving Equation (E-3) for  $\underline{Y}$  and substituting into Equation (E-2) yields

$$\underline{\Delta}^{-1} \underline{Q}_b = \underline{C} \underline{X} + \underline{D}_a \underline{Q}_a + \underline{D}_b \underline{Q}_b \quad (\text{E-4})$$

Solving this equation for  $\underline{Q}_b$  yields

$$\underline{Q}_b = [\underline{\Delta}^{-1} - \underline{D}_b]^{-1} \underline{C} \underline{X} + [\underline{\Delta}^{-1} - \underline{D}_b]^{-1} \underline{D}_a \underline{Q}_a \quad (\text{E-5})$$

Substituting this result into Equation (E-1) yields

$$\dot{\underline{X}} = \left\{ \underline{A} + \underline{B}_b [\underline{\Delta}^{-1} - \underline{D}_b]^{-1} \underline{C} \right\} \underline{X} + \left\{ \underline{B}_a + \underline{B}_b [\underline{\Delta}^{-1} - \underline{D}_b]^{-1} \underline{D}_a \right\} \underline{Q}_a \quad (\text{E-6})$$

\* The term

$$[\underline{\Delta}^{-1} - \underline{D}_b]^{-1} \quad (\text{E-7})$$

can be written as

$$[\Delta^{-1} - D_b]^{-1} \Delta^{-1} \Delta \quad (E-8)$$

or

$$\{\Delta[\Delta^{-1} - D_b]\}^{-1} \Delta \quad (E-9)$$

But, this expression reduces to

$$[I - \Delta D_b]^{-1} \Delta \quad (E-10)$$

Therefore, Equation (E-6) reduces to

$$\dot{\underline{X}} = \{A + B_b[I - \Delta D_b]^{-1} \Delta C\} \underline{X} + \{B_a + B_b[I - \Delta D_b]^{-1} \Delta D_a\} \underline{u}_a \quad (E-11)$$

The output equations,  $\underline{Y}$ , are obtained by substituting Equation (E-5) into Equation (E-2) and rearranging. The result is

$$\underline{Y} = \{C + D_b[I - \Delta D_b]^{-1} \Delta C\} \underline{X} + \{D_a + D_b[I - \Delta D_b]^{-1} \Delta D_a\} \underline{u}_a \quad (E-12)$$

The existence of the inverse of  $(I - \Delta D_b)$  is assured by considering a linear time-invariant system. If  $(I - \Delta D_b)^{-1}$  does not exist, then the system does not have a characteristic equation. But, every linear system has a characteristic equation.

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE State Space Application to System Design			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Thesis			
5. AUTHOR(S) (Last name, first name, initial) MOCK, Sanford N.			
6. REPORT DATE June 1968	7a. TOTAL NO. OF PAGES 114	7b. NO. OF REFS 21	
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)		
b. PROJECT NO.			
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
d.			
10. AVAILABILITY/LIMITATION NOTICES <del>This document is subject to special report controls and each transmission to foreign governments or foreign nationals may be made only with prior approval of the Naval Postgraduate School.</del>			
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## 13. ABSTRACT

The availability of digital and hybrid computers has led to the development of the state space approach and optimization theory for the analysis and design of control systems, particularly in space oriented problems where meaningful cost criteria can be defined. In this thesis optimization theory is investigated as applied to classical control systems, such as regulators, to determine if these techniques may be used in the design of systems to meet conventional performance standards. As part of this investigation a method has been developed which yields the overall state equations for a system from the state equations of the individual components. Also, since optimal designs are usually non-linear and time varying, a discussion of stability criteria for these systems is included.

### KEY WORDS

LINK A

## LINK B

LINK C

ROLE

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## State equations

## Optimal system

Lyapunov









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